Reconstruction of piecewise constant functions from X-ray data

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X-ray tomography

Let M be a compact *nontrapping* (any geodesic meets the boundary in finite time) Riemannian surface with *strictly convex boundary* (the second fundamental form of ∂M in M is positive definite).
The ray transform $\mathcal{I}f$ of a function $f : M \to \mathbb{R}$ is the collection of the integrals of f over maximal geodesics. It has been conjectured that the X-ray transform is injective on compact nontrapping Riemannian manifolds with strictly convex boundary. This result has been proved for piecewise constant functions in [1]. The following problem arises then:

Problem

Can we recover a piecewise constant function from its X-ray transform on a compact nontrapping Riemannian surface with strictly convex boundary ?

Main result

The problem has a positive answer if the tiling and the metric are known. The reconstruction uses specific variations through geodesics.
If the manifold is simple and the tiling is geodesic the Jacobi field associated to the variations are sufficient in the reconstruction.

A manifold is said to be *simple* if it is simply connected, if it has strictly convex boundary and has no conjugate points.

Method

1. Reconstruction in a corner by reduction of the Riemannian case to the Euclidean case in a plane
2. Reconstruction near the boundary
3. Local argument at level sets of a strictly convex foliation
4. Iteration of the local argument

Definitions

We call *triangle* on M the image of a C^1-embedding of the 2-simplex from $\mathbb{R}^3$ to M. A *tiling* of M is a finite collection of triangles $(\Delta_i)_{i\in I}$ covering M such that $\mathrm{int}(\Delta_i)$ and $\mathrm{int}(\Delta_j)$ with $i \neq j$ can intersect each other only at a common vertex or all along a common edge. A function $f : M \to \mathbb{R}$ is *piecewise constant* if there is a tiling such that f is constant on the interior of each triangle and vanishes on the edges.
If Δ is a triangle and $p \in \Delta$, the *tangent cone* of Δ at p, denoted by $C_p\Delta$, is the set $\{\dot\gamma(0), \gamma \in \mathcal{C}\} \subset T_pM$ with $\mathcal{C}$ the set of all C^1-curves starting at p staying in Δ.
Let f be a piecewise constant function, $\Delta_1, \dots, \Delta_N$ be the tiles that contains p and $a_1, \dots, a_N$ the respective values of f on those tiles. The *tangent function* of f at p is the function $T_pf : T_pM \to \mathbb{R}$ defined by

$$T_pf(u) = \begin{cases} a_i & \text{if } u \in \mathrm{int}(\Delta_i) \\ 0 & \text{if } u \notin \bigcup_{i=1}^N C_p\Delta_i \end{cases}. \qquad (1)$$

Given an orthonormal basis (ω, ν) of T_pM, we parametrize a unit tangent vector in polar coordinates by its angle $\theta \in (-\pi, \pi]$ and we denote it w_θ.

Reconstruction in a corner

Let $\Sigma \subset M$ be a strictly convex hypersurface, $p \in \Sigma$ be a vertex of the tiling. Denote ν the inward pointing normal to Σ at p and let (ω, ν) be an orthonormal basis of T_pM. Denote $H_\pm = \{w \in T_pM, \pm w \cdot \nu > 0\}$, define $E = \{\Delta, C_p\Delta \subset H_+ \cup \{0\}\}$ and enumerate the edges of tiles of E from 1 to $N+1$ by increasing angles.

Define the corner C as the union of tiles of E and S the sector defined by $S := \bigcup_{\Delta\in E} C_p\Delta$.
Let γ_θ be the geodesic starting from p with initial velocity w_θ and denote ℓ_θ the line defined by $y = \frac{1}{\cos(\theta)} + \tan(\theta)x$.
Let $\Gamma_\theta : (-\varepsilon_0, \varepsilon_0) \times [a, b] \to M$ be a variation through geodesics starting from γ_θ. Denote J_θ its Jacobi field and $\gamma_\theta^\varepsilon = \Gamma_\theta(\varepsilon, \cdot)$.

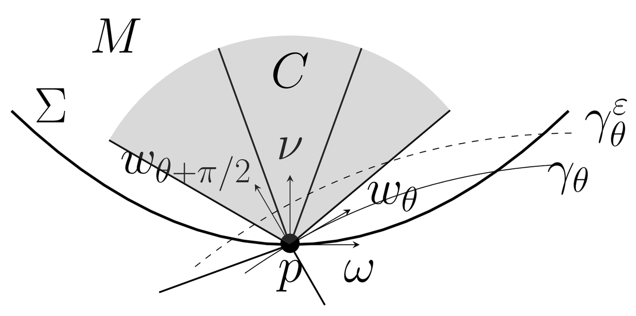

Figure 1:The corner and the variation.

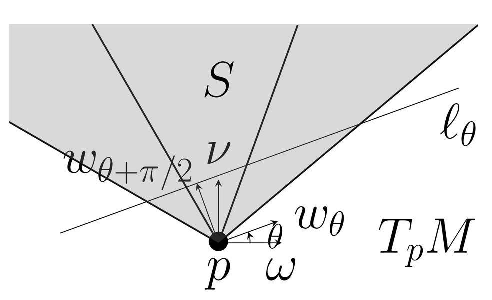

Figure 2:Tangent data.

Lemma 1: Reduction to the Euclidean case (Ilmavirta, Lehtonen, Salo)

If $J_\theta(0) = w_{\theta+\pi/2}$, then for all θ near 0 we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\gamma_\theta^\varepsilon \cap C} f\, ds = \int_{\ell_\theta \cap S} T_pf\, ds =: If(\theta). \qquad (2)$$

Taking $N-1$ derivatives in θ allows us to reduce the reconstruction to the inversion of a Vandermonde matrix

$$\begin{pmatrix} \alpha_1 - \alpha_2 & \dots & \alpha_N - \alpha_{N+1} \\ \vdots & & \vdots \\ \alpha_1^N - \alpha_2^N & \dots & \alpha_N^N - \alpha_{N+1}^N \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} F(0) \\ \vdots \\ F^{(N-1)}(0) \end{pmatrix} \qquad (3)$$

where $\alpha_i = 1/\tan(\theta_i)$ if θ_i the angle associated to an edge e_i in the corner and $F(t) = \cos^2(\arctan(t)) If(\arctan(t))$.

Near the boundary

Side tiles: (see fig. 3) We have that

$$a_i = \lim_{\varepsilon\to 0} \frac{\kappa}{2\,\varepsilon} \int_{\gamma_i^\varepsilon} f\, ds, \qquad (4)$$

where a_i is the value of f on Δ_i, κ is the curvature of ∂M at p and γ_i^ε starting from p with initial velocity $\pm\omega + \varepsilon\nu$.

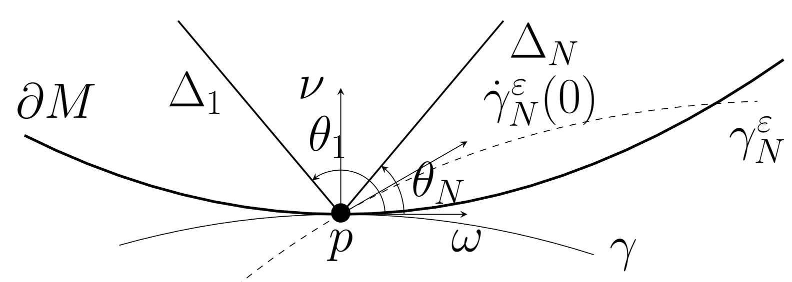

Figure 3:Side tiles

Corner tiles: We recover the values thanks to the previous section with the same variation as in [1]. Let $w_\theta(\varepsilon)$ be the unit vector defined as the parallel transport of w_θ along the geodesic through $w_{\theta+\pi/2}$ by distance ε. Define $\gamma_\theta^\varepsilon$ as the maximal geodesic through $w_\theta(\varepsilon)$. We have

$$\int_{\gamma_\theta^\varepsilon \cap C} f\, ds = \int_{\gamma_\theta^\varepsilon} f\, ds - a_1(t_1(\varepsilon) - t_0(\varepsilon)) - a_N(t_{N+1}(\varepsilon) - t_N(\varepsilon)) \qquad (5)$$

with for the meeting times with edges $i = 1, N$,

$$t_i'(0) = \frac{\cos(\theta_i - \theta)}{\sin(\theta_i - \theta)}, \qquad (6)$$

for the ending times $i \in \{0, N+1\}$, if $\theta \neq 0$

$$t_i'(0) = -\frac{J_\theta(t_i(0)) \cdot \nu}{\dot\gamma_\theta(t_i(0)) \cdot \nu}, \qquad (7)$$

and if $\theta = 0$,

$$t_{N+1}(\varepsilon) \underset{\varepsilon\to 0^+}{\sim} \sqrt{\frac{2\,\varepsilon}{\kappa}} \qquad (8)$$

(with the opposite sign for t_0).

Acknowledgement

I would like to thank M. Salo for bringing this problem to my attention, for having closely followed its resolution and for our helpful discussions.

Inside the manifold

There exist a strictly convex function $\varphi : M \to \mathbb{R}$ so M is foliated by the level sets $\{\varphi = c\}$ for $\min\varphi < c \leq \max\varphi$ which are strictly convex hypersurfaces. Let p be a vertex on the level set $\Sigma := \{\varphi = c\}$.

Lemma 2: Local reconstruction

If the values of f are known in $\{\varphi > c\}$, then one can reconstruct the values of f near p from the full knowledge of the tiling and the metric.

There are three mutually exclusive types of tiles having p as a vertex:

1. simplices Δ such that $C_p\Delta \cap H_- \neq \varnothing$,
2. simplices Δ such that $C_p\Delta \subset H_+ \cup T_p\Sigma$ and $C_p\Delta \cap T_p\Sigma \neq \{0\}$, and
3. simplices Δ such that $C_p\Delta \subset H_+ \cup \{0\}$ (the corner).

General case: The method is a direct adaptation of the injectivity proof: the full knowledge of the tiling allows us to compute integrals of f on each tile on which we know its value. The first simplices always intersect $\{\varphi > c\}$. The second ones are recovered with the same geodesics as the side tiles on the boundary. The third ones are recovered thanks to the corner reconstruction section.

Simple manifolds & geodesic tilings: Suppose now that the manifold is simple and the tiling is *geodesic* (edges are geodesic segments or boundary segments). The first and second type simplices intersect $\{\varphi > c\}$ because Σ is strictly convex.
Most importantly, we can explicit $\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\gamma_\theta^\varepsilon \cap C} f\, ds$, used to reconstruct f in the corner. To do so, the variation through geodesics $\Gamma(\varepsilon, t) = \gamma_\theta^\varepsilon(t)$ has to satisfy that its Jacobi field never vanishes. Since the manifold is simple, one can always find one along γ_θ which never vanishes and has value $w_{\theta+\pi/2}$ at p.

Lemma 3: Parametrization of integrals

There exist $\varepsilon_1 > 0$, $N \in \mathbb{N}$, C^1-functions $t_i : [0, \varepsilon_1) \to [a, b]$ for $i \in \{1, \dots, N+1\}$ and $(a_i)_{i\in\{1,\dots,N\}}$ values of f associated to tiles Δ_i such that,

$$\forall \varepsilon \in (0, \varepsilon_1), \int_{\gamma_\theta^\varepsilon} f\, ds = \sum_{i=1}^N a_i(t_{i+1}(\varepsilon) - t_i(\varepsilon)). \qquad (9)$$

Sketch of proof: The parametrizations of the meeting times are obtained by the implicit function theorem applied at each meeting point of γ_θ with an edge of the tiling. This is allowed by the fact that there are no tangential edges other than segments of γ_θ because the tiling is geodesic. We also obtain an explicit formula of the derivatives of times.

Global reconstruction

Denote by $c_1 > \dots > c_K$ the distinct elements of the set $\{\max_\Delta \varphi, \Delta \text{ triangle of the tiling}\}$.
Since $c_1 = \max\varphi$ and $\{\varphi = \max\varphi\} \subset \partial M$, we can reconstruct f on the simplices Δ such that $\max_\Delta \varphi = c_1$ and hence in the set $\{\varphi > c_2\}$.
The lemma 2 allows us to reconstruct f near the level set $\{\varphi = c_2\}$ and therefore on each tile Δ_i such that $\max_{\Delta_i} \varphi = c_2$. Iterating this method, we reach the index K and we reconstruct f everywhere on M.

References

[1] J. Ilmavirta, J. Lehtonen, and M. Salo.
Geodesic X-ray tomography for piecewise constant functions on nontrapping manifolds.
2017.