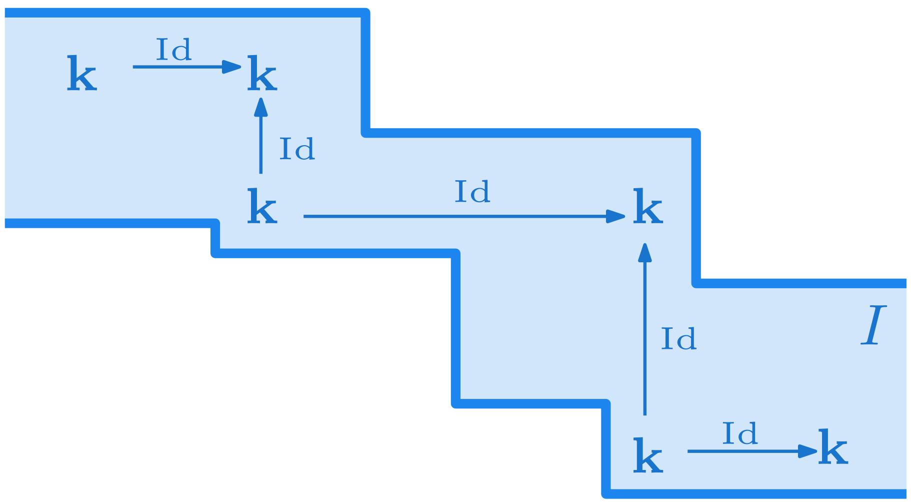
Local characterization of decomposability for 2-parameter persistence modules

Vadim Lebovici

arxiv:2008.02345
arxiv:2002.08894

joint work with M. Botnan and S. Oudot

(adapted from S. Oudot's slides)



Interactions between representation theory and topological data analysis
Center for Advanced Study, Oslo – 22/12/04

Persistence modules

$(P, \leq)$ a poset, $\mathbf{k}$ a field

finite dimensional $\mathbf{k}$-vector spaces

Persistence module : functor $M : P \to \mathsf{vec}_{\mathbf{k}}$

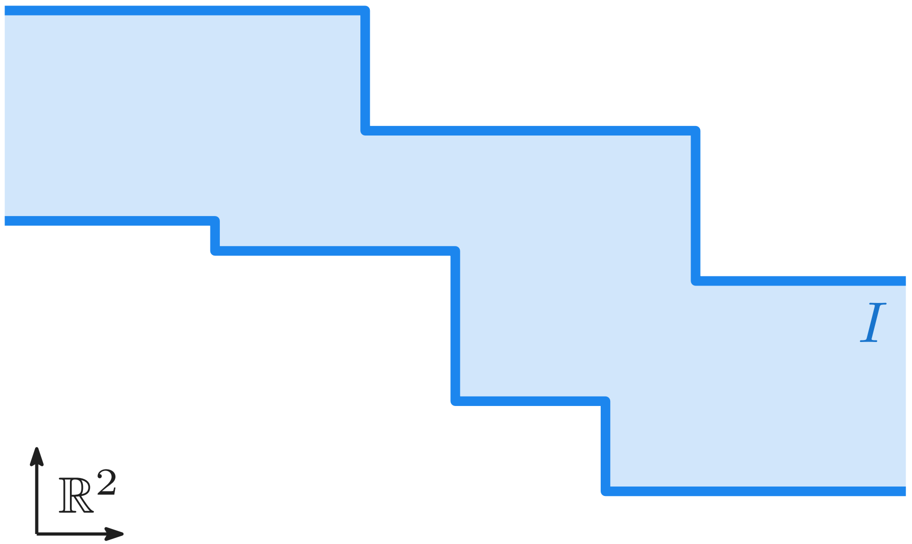
Interval : $I \subseteq P$ that is :

Not. : $I \in \mathrm{Int}(P)$

- convex $(s, t \in I \implies u \in I \;\forall\; s \leq u \leq t)$
- connected $(s, t \in I \implies \exists \{u_i\}_{i=0}^{r} \subseteq I$ s.t. $s = u_0 \leq u_1 \geq \cdots \geq u_r = t)$

Ex. : (i) P totally ordered

(ii) $P = \mathbb{R}^2$ with coordinatewise order



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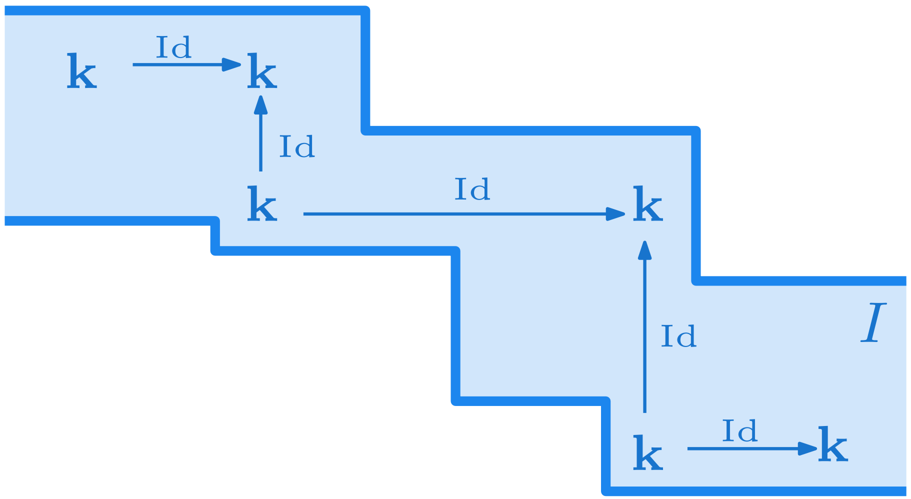
Interval module : indicator module $\mathbf{k}_I$ of an interval $I \subseteq P$

$$\mathbf{k}_I(t) = \begin{cases} \mathbf{k} \text{ if } t \in I \\ 0 \text{ otherwise} \end{cases}$$

$$\mathbf{k}_I(s \leq t) = \begin{cases} \mathrm{Id}_{\mathbf{k}} \text{ if } s, t \in I \\ 0 \text{ otherwise} \end{cases}$$



Rq : $\mathrm{End}(\mathbf{k}_I) \simeq \mathbf{k}$ → indecomposable

Persistence modules

Interval modules are **described by their support** :

- geometric descriptor
- efficient to encode (small / simple dictionary)
- readily interpretable
- easy to vectorize (for machine learning)

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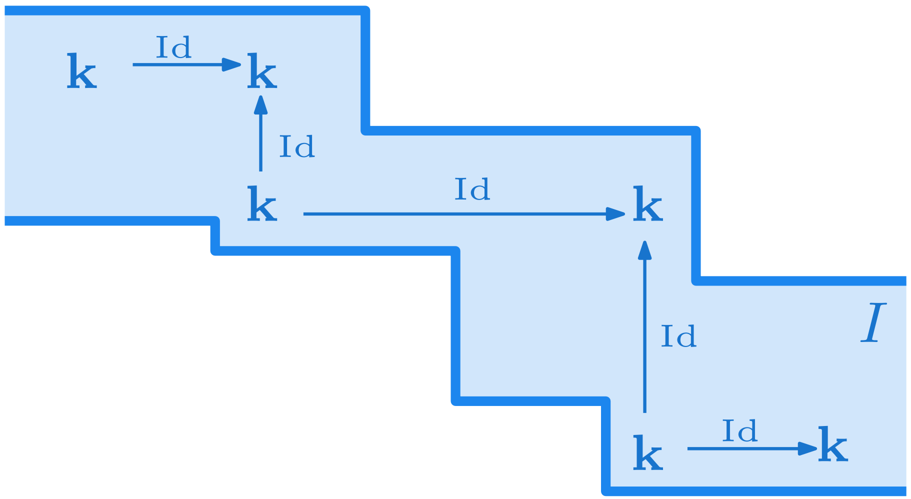

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Decomposition of persistence modules

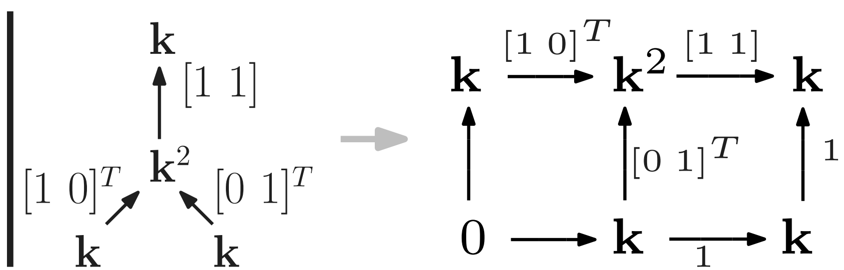
Thm. (Crawley-Boevey '15, Botnan, Crawley-Boevey '20)
Assume P is totally ordered. Then, $M \simeq \bigoplus_{I \in \mathcal{I}} \mathbf{k}_I$ where the I's are intervals of P.

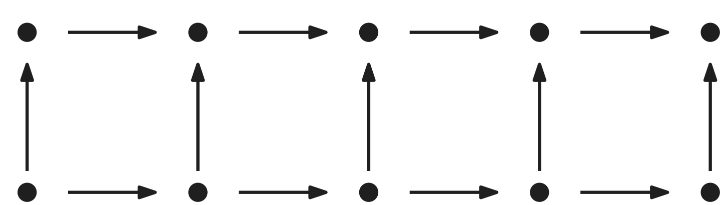
Que. What about $P = X_1 \times \cdots \times X_n \subseteq \mathbb{R}^n$ with $n \geq 2$?

Thm. (Botnan, Crawley-Boevey '20) $(P, \leq)$ any poset

$$M \simeq \bigoplus_{\alpha \in A} M_\alpha \quad \text{where the } M_\alpha\text{'s are indecomposable}$$

Pbs. (i) non-thin indecomposables (ii) wild type posets



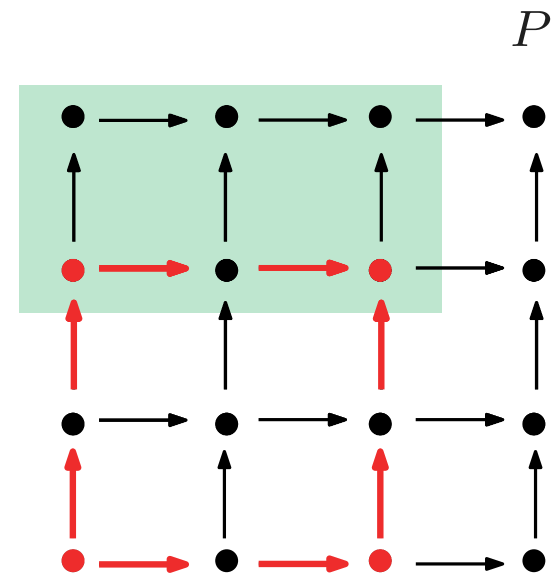


Questions

- characterize datasets/filtrations whose modules are interval-dec.
- given a filtration : check interval-dec. & extract summands

Local characterizations faster than with a full decomposition (e.g. `MeatAxe`)

Local characterizations



Poset : $P = X \times Y \subseteq \mathbb{R}^2$

(i) **Collection of supports** : $\mathcal{S} \subseteq \mathrm{Int}(X \times Y)$

Ex. rectangles $\mathcal{S} = \mathrm{Rec}(X \times Y)$

$$= \{I \times J : (I, J) \in \mathrm{Int}(X) \times \mathrm{Int}(Y)\}$$

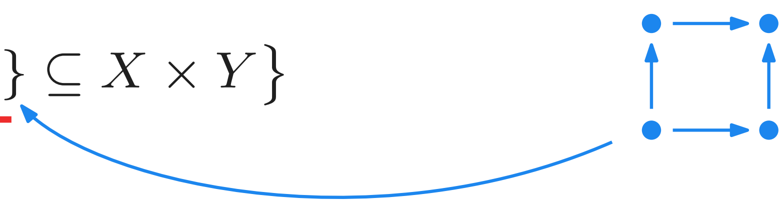
associated **$\mathcal{S}$-decomposable** modules :

$$\langle \mathcal{S} \rangle = \left\{ M \text{ pers. mod. } : M \simeq \bigoplus_{S \in \mathcal{S}} \mathbf{k}_S^{m_S},\ m_S \in \mathbb{Z} \right\}$$

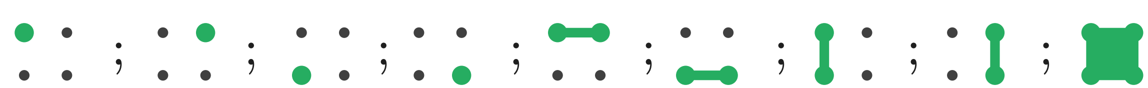
(ii) **Test subsets** : collection $\mathcal{Q}$ of subsets of $X \times Y$

For $Q \in \mathcal{Q}$, $\quad \mathcal{S}_{|Q} := \{S \cap Q : S \in \mathcal{S}\} \subseteq \mathrm{Conv}(X \times Y) \longrightarrow \langle \mathcal{S}_{|Q} \rangle$

Ex. $\mathcal{Q} = \{\text{squares of } X \times Y\} = \big\{\{x, x'\} \times \{y, y'\} \subseteq X \times Y\big\}$



For any square Q of $X \times Y$,

$\mathrm{Rec}(X \times Y)_{|Q} = \mathrm{Rec}(Q) = \{$  $\}$

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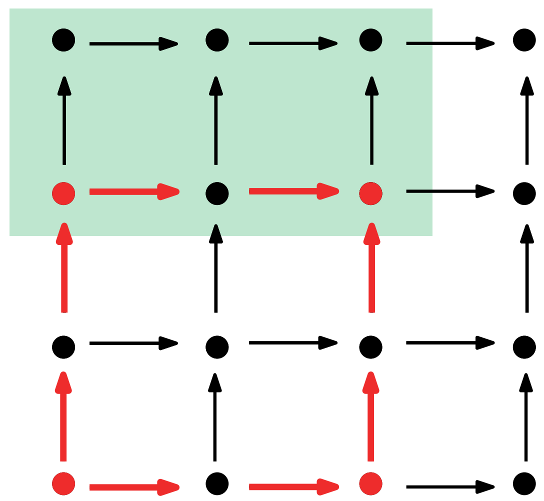

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Rk. $\mathcal{S}$-decomposability is closed under taking restrictions, i.e.

$$M \in \langle \mathcal{S} \rangle \implies \forall Q \in \mathcal{Q},\, M_{|Q} \in \langle \mathcal{S}_{|Q} \rangle$$

Main question

Identify $\mathcal{S}$ and $\mathcal{Q}$ such that : $\quad M \in \langle \mathcal{S} \rangle \impliedby \forall Q \in \mathcal{Q},\, M_{|Q} \in \langle \mathcal{S}_{|Q} \rangle$

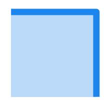
Known local characterizations

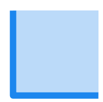
Main question

Identify $\mathcal{S}$ and $\mathcal{Q}$ such that : $\quad M \in \langle \mathcal{S} \rangle \quad \Longleftarrow \quad \forall Q \in \mathcal{Q},\ M_{|Q} \in \langle \mathcal{S}_{|Q} \rangle$

Setting : $\mathcal{S} = \mathsf{Blc}(X \times Y) \qquad \mathcal{Q} = \{\text{squares of } X \times Y\}$

$\mathsf{Blc}(X \times Y) = \{$


"death quadrants"

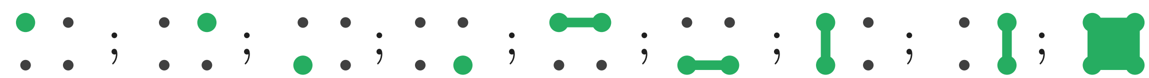

"birth quadrants"

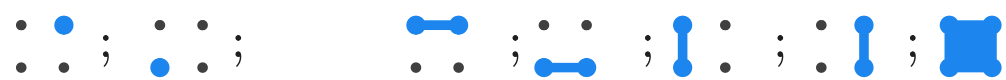

"horizontal bands"

"vertical bands"

$\}$

Rk. For any square Q,

$\mathsf{Rec}(X \times Y)_{|Q} = \{$  $\}$

$\mathsf{Blc}(X \times Y)_{|Q} = \{$  $\}$

Thm. (Cochoy, Oudot '20) $\quad \mathcal{S} = \mathsf{Blc}(X \times Y) \qquad \mathcal{Q} = \{\text{squares of } X \times Y\}$

M is block-dec. iff its restriction to any square is block-dec., i.e.

$$M \in \langle \mathsf{Blc}(X \times Y) \rangle \quad \Longleftrightarrow \quad \forall Q \in \mathcal{Q},\ M_{|Q} \in \langle \mathsf{Blc}(Q) \rangle$$

Silly question

Case : $\mathcal{S} = \mathsf{Int}(X \times Y)$ and $\mathcal{Q} = \{\text{totally ordered subsets of } X \times Y\}$

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Case : $\mathcal{S} = \mathsf{Int}(X \times Y)$ and $\mathcal{Q} = \{$totally ordered subsets of $X \times Y\}$

Fact. Interval-dec. cannot be checked on totally ordered subsets.

Proof. restrictions to totally ordered subsets are **always** interval-dec (Crawley-Boevey).

$$M := \begin{array}{ccccc} \mathbf{k} & \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & \mathbf{k}^2 & \xrightarrow{[1\ 1]} & \mathbf{k} \\ {\scriptstyle 0}\uparrow & & {\scriptstyle \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}\uparrow & & {\scriptstyle 1}\uparrow \\ 0 & \xrightarrow{0} & \mathbf{k} & \xrightarrow{1} & \mathbf{k} \end{array}$$

→ indecomposable

→ not an interval module

Interval-decomposability is not local

Case : $\mathcal{S} = \mathsf{Int}(X \times Y)$ and $\mathcal{Q} = \{$finite strict subgrids of $X \times Y\}$

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Thm. (Botnan, L., Oudot '20)

Assume that $\#X > 2$ and $\#Y > 2$. Let $2 \leq m < \min(\#X, \#Y)$.

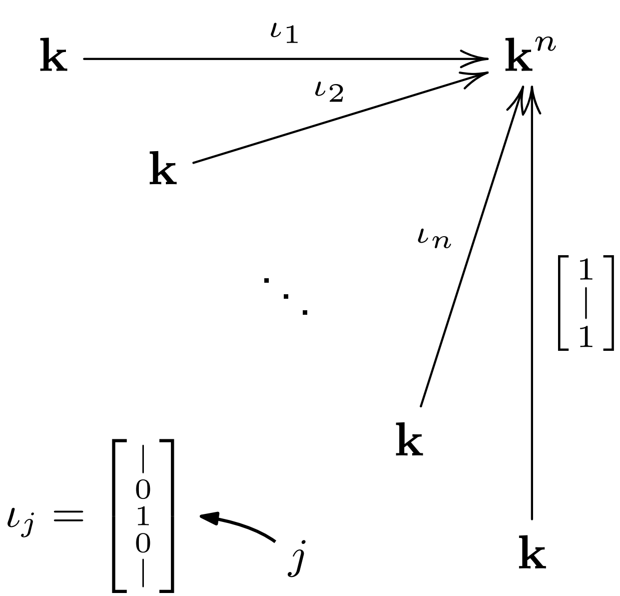
There exists a module M over $X \times Y$ such that :

(i) M is not interval-dec.

(ii) $M_{|Q}$ is interval-dec for any grid $Q \subset X \times Y$ of side-length at most m.

Proof :

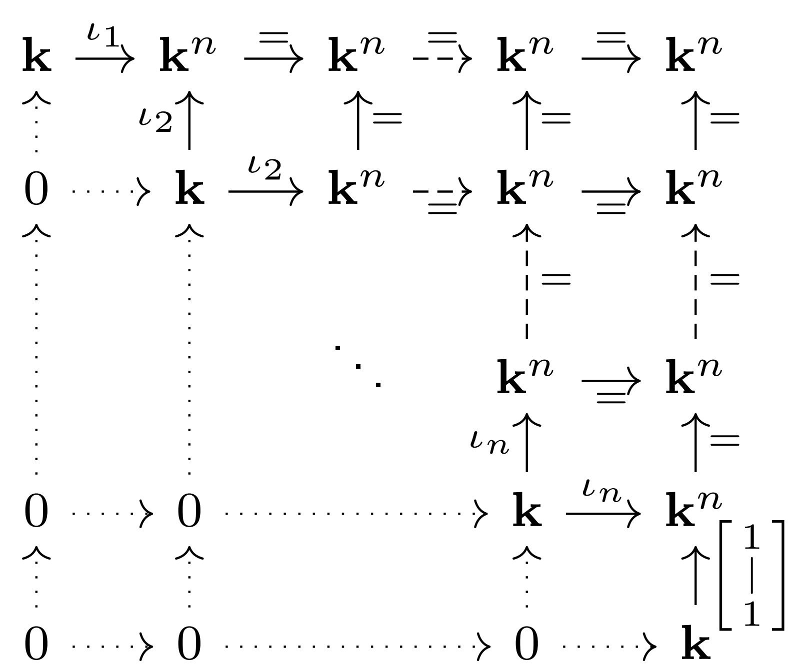
Adapt the indecomposable :



$\longrightarrow$ $M :=$

indec.

not interval module



Rectangle-decomposability is local

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Algebraic characterization

$$\begin{array}{ccc} M(c) & \xrightarrow{\delta} & M(d) \\ \beta \uparrow & & \uparrow \gamma \\ M(a) & \xrightarrow{\alpha} & M(b) \end{array}$$

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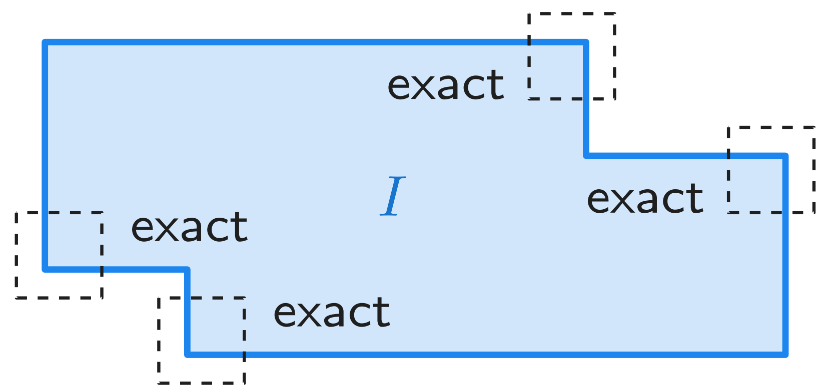

Exactness condition : $M(a) \xrightarrow{\varphi = (\alpha, \beta)} M(b) \oplus M(c) \xrightarrow{\psi = \gamma - \delta} M(d)$ is exact.

Ex. Interlevel sets persistent homology

$$(x, y) \in \mathbb{R}^{\mathsf{op}} \times \mathbb{R} \mapsto H_k\left(f^{-1}(x, y)\right)$$

Ex. exact

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbf{k} & \longrightarrow & 0 \end{array}$$

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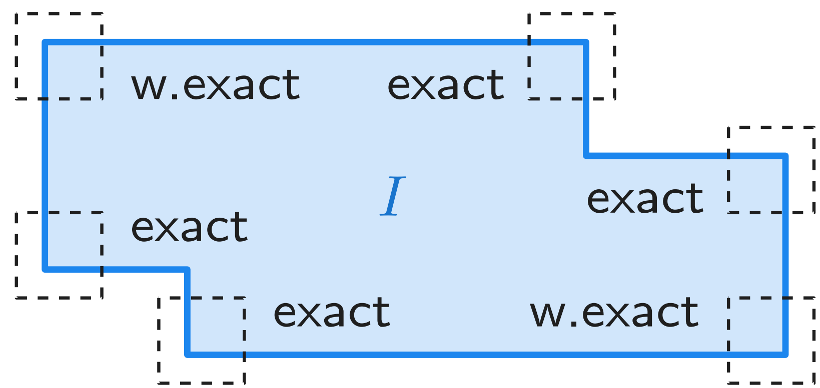

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Exactness condition : $M(a) \xrightarrow{\varphi = (\alpha, \beta)} M(b) \oplus M(c) \xrightarrow{\psi = \gamma - \delta} M(d)$ is exact.

Weak exactness condition :

$$\text{Im}\ \rho = \text{Im}\ \gamma \cap \text{Im}\ \delta$$

$$\text{Ker}\ \rho = \text{Ker}\ \alpha + \text{Ker}\ \beta$$

Ex.

exact

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ \mathbf{k} & \longrightarrow & 0 \end{array}$$

w.exact

$$\begin{array}{ccc} \mathbf{k} & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

not w.exact

$$\begin{array}{ccc} \mathbf{k} & \longrightarrow & 0 \\ = \uparrow & & \uparrow \\ \mathbf{k} & \xrightarrow[=]{} & \mathbf{k} \end{array}$$

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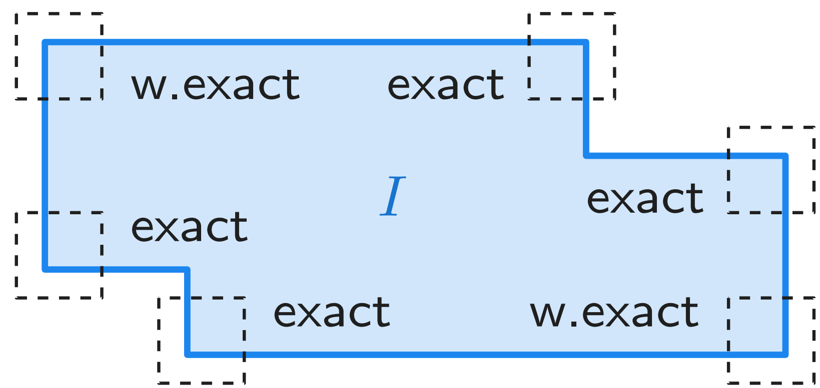

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Lem.

M is exact $\Leftrightarrow M_{|Q} \in \langle \mathrm{Blc}(Q) \rangle$

M is w. exact $\Leftrightarrow M_{|Q} \in \langle \mathrm{Rec}(Q) \rangle$

for all squares Q

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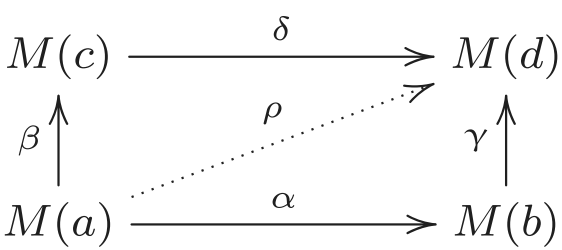
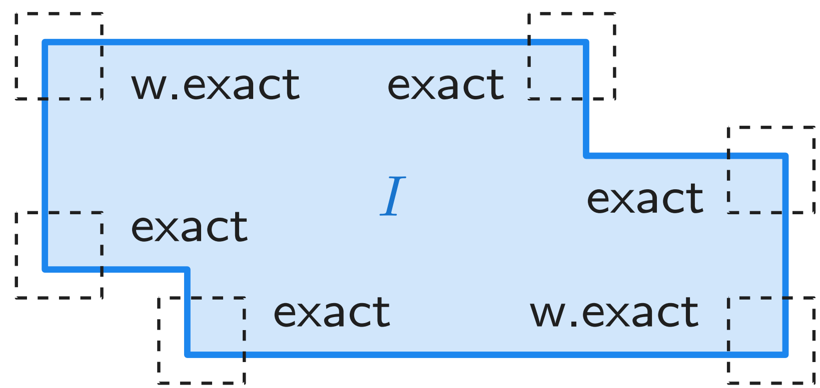
$$M \in \langle \text{Rec}(X \times Y) \rangle \iff \underline{M \text{ is weakly exact}}$$

Algebraic characterization





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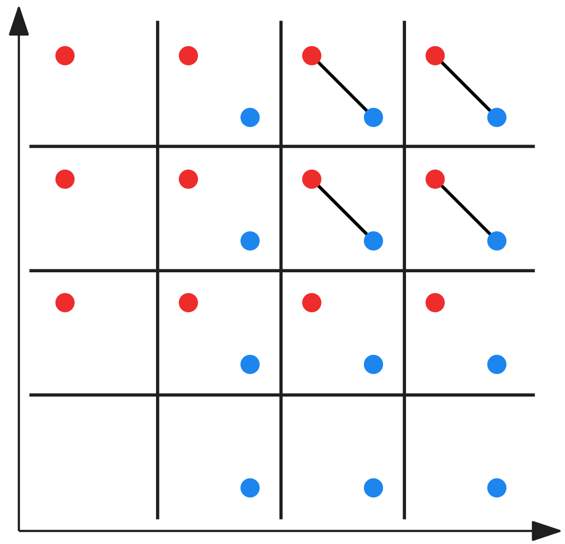
Application : checking rectangle-dec.

$P = \{1, \ldots, n\} \times \{1, \ldots, m\} \subset \mathbb{R}^2$



Given $F : P \to$ Simp (finite simplicial complexes) :

Goal :

- determine whether $H_k F$ is rectangle-dec.
- if so, compute the decomposition of $H_k F$

Straightforward approach

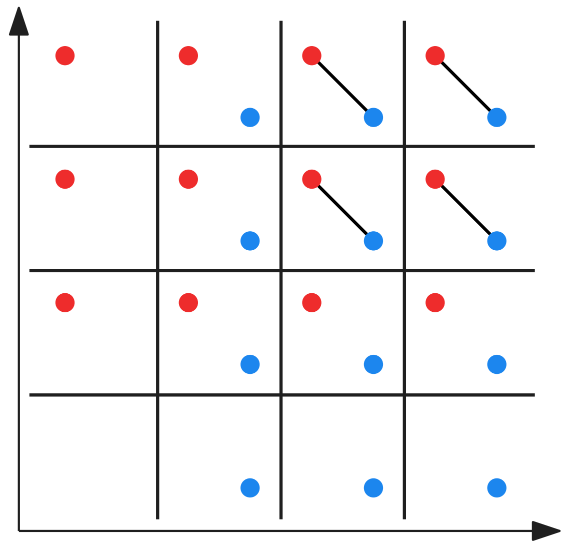
- compute direct-sum decomposition ⟶ $O\left(r^{2\omega+1}\right)$ (Dey, Xin '19)
- check summands one by one (thin-ness and support)

Time complexity : $O\left(r^{2\omega+1}\right)$

r : total number of simplices in F

$\omega \approx 2.373$: exponent for matrix multiplication

Application : checking rectangle-dec.



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Given $F : P \to$ Simp (finite simplicial complexes) :

Goal :

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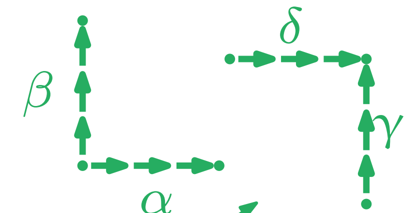
Optimized approach for rectangle-dec.



(i) compute rank invariant rk : $P \times P \to \mathbb{N}$ ⟶ $O(r^4)$

(ii) compute all $\dim(\mathrm{Im}\,\gamma \cap \mathrm{Im}\,\delta)$ and $\dim(\mathrm{Ker}\,\alpha + \mathrm{Ker}\,\beta)$

Method : compute $2 \times r^2$ zigzag barcodes in $O(r^\omega)$ each ⟶ $O\left(r^{2+\omega}\right)$

(iii) check weak exactness, i.e. $\dim \mathrm{Im}\,\rho = \dim(\mathrm{Im}\,\gamma \cap \mathrm{Im}\,\delta)$ ⟶ $O(r^4)$

$\dim \mathrm{Ker}\,\rho = \dim(\mathrm{Ker}\,\alpha + \mathrm{Ker}\,\beta)$

Time complexity : $O\left(r^{2+\omega}\right)$

r : total number of simplices in F

$\omega \approx 2.373$: exponent for matrix multiplication

Application : Pyramid basis theorem

Setting : $f : X \to \mathbb{R}$ of "Morse type"

i.e. with "critical values" $a_0 = -\infty < a_1 < \dots < a_n < a_{n+1} = +\infty$

s.t. $f^{-1}(a_i, a_{i+1}) \simeq Y_i \times (a_i, a_{i+1})$

$$\begin{array}{ccc} f^{-1}(a_i,a_{i+1}) & \simeq & Y_i \times (a_i, a_{i+1}) \\ & \overset{f}{\searrow} & \downarrow \text{projection} \\ & & (a_i, a_{i+1}) \end{array}$$

Consider $-\infty < s_0 < a_1 < \cdots < a_n < s_n < +\infty$. We have inclusions :

$$\begin{array}{ccccccccccc}
 & & & & (X_0^2, X_0^2) & \to & (X_0^2, {}_2^2X) & & & & \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & (X_0^1, X_0^1) & \to & (X_0^2, X_0^1) & \to & (X_0^2, {}_1^2X) & \to & (X_0^2, {}_1^1X) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
(X_0^0, X_0^0) & \to & (X_0^1, X_0^0) & \to & (X_0^2, X_0^0) & \to & (X_0^2, {}_0^2X) & \to & (X_0^2, {}_0^1X) & \to & (X_0^2, {}_0^0X) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
X_0^0 & \longrightarrow & X_0^1 & \longrightarrow & X_0^2 & \longrightarrow & (X_0^2, X_2^2) & \to & (X_0^2, X_1^2) & \to & (X_0^2, X_0^2) \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & X_1^1 & \longrightarrow & X_1^2 & \longrightarrow & (X_1^2, X_2^2) & \to & (X_1^2, X_1^2) & & \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & & & X_2^2 & \longrightarrow & (X_2^2, X_2^2) & & & &
\end{array}$$

where $X_i^j := f^{-1}(s_i, s_j)$

${}_i^jX := f^{-1}(s_0, s_i) \cup f^{-1}(s_j, s_n)$

Application : Pyramid basis theorem

Setting : $f : X \to \mathbb{R}$ of "Morse type"

i.e. with "critical values" $a_0 = -\infty < a_1 < \ldots < a_n < a_{n+1} = +\infty$

s.t. $f^{-1}(a_i, a_{i+1}) \simeq Y_i \times (a_i, a_{i+1})$

$$\begin{array}{ccc} f^{-1}(a_i, a_{i+1}) \simeq & Y_i \times (a_i, a_{i+1}) & \\ {}_{f}\searrow & \downarrow & \text{projection} \\ & (a_i, a_{i+1}) & \end{array}$$

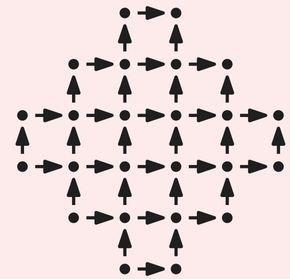
Taking (relative) homology :

$$H_p\mathcal{P} := \begin{array}{ccccccccccc}
 & & & & H_p(X_0^2, X_0^2) & \to & H_p(X_0^2, {}^2_2X) & & & & \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & H_p(X_0^1, X_0^1) & \to & H_p(X_0^2, X_0^1) & \to & H_p(X_0^2, {}^2_1X) & \to & H_p(X_0^2, {}^1_1X) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
H_p(X_0^0, X_0^0) & \to & H_p(X_0^1, X_0^0) & \to & H_p(X_0^2, X_0^0) & \to & H_p(X_0^2, {}^2_0X) & \to & H_p(X_0^2, {}^1_0X) & \to & H_p(X_0^2, {}^0_0X) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
H_p(X_0^0) & \to & H_p(X_0^1) & \to & H_p(X_0^2) & \to & H_p(X_0^2, X_2^2) & \to & H_p(X_0^2, X_1^2) & \to & H_p(X_0^2, X_0^2) \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & H_p(X_1^1) & \to & H_p(X_1^2) & \to & H_p(X_1^2, X_2^2) & \to & H_p(X_1^2, X_1^2) & & \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & & & H_p(X_2^2) & \to & H_p(X_2^2, X_2^2) & & & &
\end{array}$$

Application : Pyramid basis theorem

Thm. (Bendich, Edelsbrunner, Morozov, Patel, '13)

$$H_p\mathcal{P} \simeq \bigoplus_{\substack{R \in \mathsf{Rec}(\mathbb{Z}^2) \\ \text{s.t. } R\cap P \text{ is maximal in } P}} \mathbf{k}_{R\cap P}^{m_R} \qquad \text{where } P =$$



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H_p(X_0^0, X_0^0) & \longrightarrow & H_p(X_0^1, X_0^0) & \longrightarrow & H_p(X_0^2, X_0^0) & \longrightarrow & H_p(X_0^2, {}_0^2X) & \longrightarrow & H_p(X_0^2, {}_0^1X) & \longrightarrow & H_p(X_0^2, {}_0^0X) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
H_p(X_0^0) & \longrightarrow & H_p(X_0^1) & \longrightarrow & H_p(X_0^2) & \longrightarrow & H_p(X_0^2, X_2^2) & \longrightarrow & H_p(X_0^2, X_1^2) & \longrightarrow & H_p(X_0^2, X_0^2) \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
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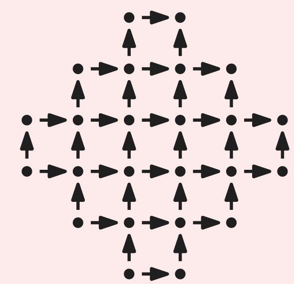

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 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
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\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
H_p(X_0^0) & \longrightarrow & H_p(X_0^1) & \longrightarrow & H_p(X_0^2) & \longrightarrow & H_p(X_0^2, X_2^2) & \longrightarrow & H_p(X_0^2, X_1^2) & \longrightarrow & H_p(X_0^2, X_0^2) \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
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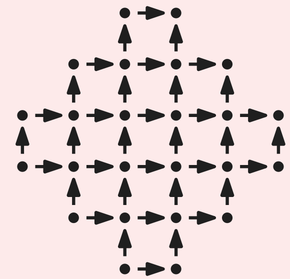

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\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
H_p(X_0^0, X_0^0) & \rightarrow & H_p(X_0^1, X_0^0) & \rightarrow & H_p(X_0^2, X_0^0) & \rightarrow & H_p(X_0^2, {}^2_0X) & \rightarrow & H_p(X_0^2, {}^1_0X) & \rightarrow & H_p(X_0^2, {}^0_0X) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
H_p(X_0^0) & \rightarrow & H_p(X_0^1) & \rightarrow & H_p(X_0^2) & \rightarrow & H_p(X_0^2, X_2^2) & \rightarrow & H_p(X_0^2, X_1^2) & \rightarrow & H_p(X_0^2, X_0^2) \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & H_p(X_1^1) & \rightarrow & H_p(X_1^2) & \rightarrow & H_p(X_1^2, X_2^2) & \rightarrow & H_p(X_1^2, X_1^2) & \rightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & & & H_p(X_2^2) & \rightarrow & H_p(X_2^2, X_2^2) & \rightarrow & 0 & \rightarrow & 0
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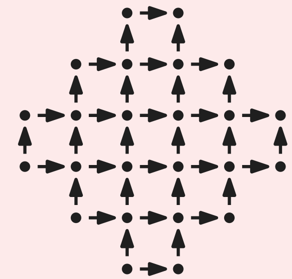

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\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
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\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
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 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
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 & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
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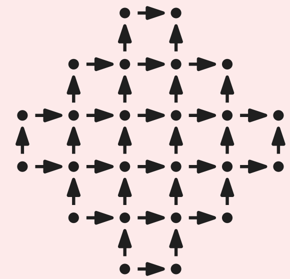
where $P =$ 

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\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
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\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
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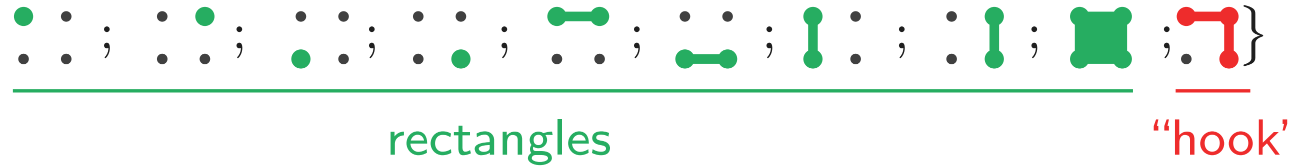
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{PB}_1 & \longrightarrow & H_p(X_1^1) & \longrightarrow & H_p(X_1^2) & \longrightarrow & H_p(X_1^2, X_2^2) & \longrightarrow & H_p(X_1^2, X_1^2) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{PB}_3 & \longrightarrow & \mathrm{PB}_2 & \longrightarrow & H_p(X_2^2) & \longrightarrow & H_p(X_2^2, X_2^2) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Rectangle-dec. is the biggest local class

Case : $\mathcal{S} \subseteq \text{Int}(X \times Y)$ s.t. for all square Q in $X \times Y$:

$\mathcal{S}_{|Q} \supseteq \{$  $\}$

$\mathcal{Q} = \{\text{squares of } X \times Y\}$

Rk. $\text{Int}(X \times Y)_{|Q} = \{$  $\}$

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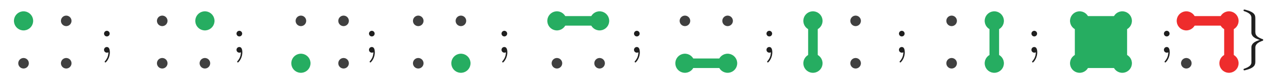
$\mathcal{S}_{|Q} \supseteq \{$  $\}$

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Thm. (Botnan, L., Oudot '20)

Assume that $\#X \geq 2$ and $\#Y \geq 2$ but $(\#X, \#Y) \neq (2,2)$.

There exists a module M over $X \times Y$ such that :

(i) M is not $\langle \mathcal{S} \rangle$-dec.

(ii) $M_{|Q}$ is $\langle \mathcal{S}_{|Q} \rangle$-dec for any square Q of $X \times Y$.

Proof :

$$\begin{array}{ccccc} \mathbf{k} & \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & \mathbf{k}^2 & \xrightarrow{[1\ 1]} & \mathbf{k} \\ \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & & \uparrow{\scriptstyle 1} \\ 0 & \xrightarrow{0} & \mathbf{k} & \xrightarrow{1} & \mathbf{k} \end{array}$$

→ indecomposable not interval module

→ $\langle \mathcal{S}_{|Q} \rangle$-decomposable for all square Q

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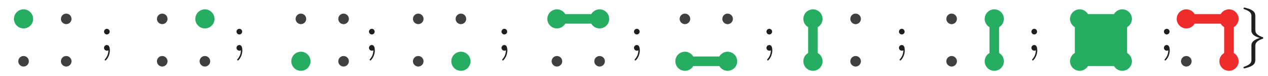
$\mathcal{S}_{|Q} \supseteq \{$  $\}$

rectangles

"hook"

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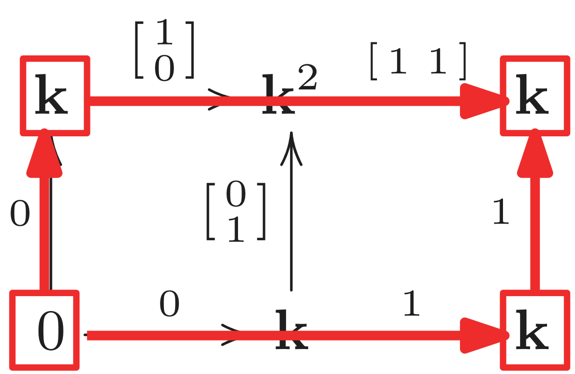

→ indecomposable not interval module

→ $\langle \mathcal{S}_{|Q} \rangle$-decomposable for all square Q

Ideas of proof : interval-decomposition
(Crawley-Boevey)

1. Define a *counting functor* for each interval I :

$$C_I : \mathrm{vec}_{\mathbf{k}}^{\mathbb{R}} \to \mathrm{vec}_{\mathbf{k}}$$

$$M \mapsto \mathbf{k}^{\mathrm{mult}(\mathbf{k}_I; M)} \qquad \text{where } \mathrm{mult}(\mathbf{k}_I; M) := \max\{n \mid M \simeq \mathbf{k}_I^n \oplus N\}$$

2. Define a submodule M_I of M for each interval I such that :

$$M_I \simeq \mathbf{k}_I^{\mathrm{mult}(\mathbf{k}_I; M)}$$

3. Show that $M = \bigoplus_I M_I$

   - show that the M_I's are in direct sum
   - show that the sum of the M_I's covers M

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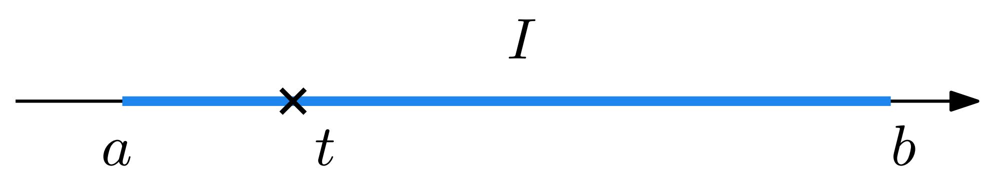
Details : For $I = (a, b)$:



- $\mathrm{Im}_I^+(t) := \bigcap_{a<s\leq t} \mathrm{Im}\, M(s \to t)$ (elements alive at least since a and still at t)

- $\mathrm{Im}_I^-(t) := \sum_{s\leq a} \mathrm{Im}\, M(s \to t)$ (elements born before a and still alive at t)

  ↳ $\mathrm{Im}_I^+(t)/\mathrm{Im}_I^-(t)$ (elements alive at t that were born at a)

- $\mathrm{Ker}_I^+(t) := \bigcap_{s\geq b} \mathrm{Ker}\, M(t \to s)$ (elements alive at t but not after b)

- $\mathrm{Ker}_I^-(t) := \sum_{t\leq s<b} \mathrm{Ker}\, M(s \to t)$ (elements alive at t and dead before b)

  ↳ $\mathrm{Ker}_I^+(t)/\mathrm{Ker}_I^-(t)$ (elements alive at t that die at b)

Ideas of proof : interval-decomposition
(Crawley-Boevey)

1. Define a *counting functor* for each interval I :

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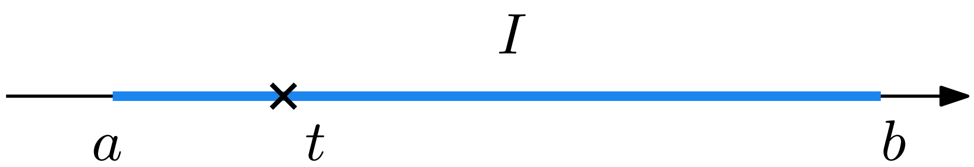

$$C_I[t] := \left(\mathrm{Im}_I^+(t) \cap \mathrm{Ker}_I^+(t)\right) / \left(\left(\mathrm{Im}_I^+(t) \cap \mathrm{Ker}_I^-(t)\right) + \left(\mathrm{Im}_I^-(t) \cap \mathrm{Ker}_I^+(t)\right)\right)$$

(alive at least since a but not after b)

(alive since a but dead before b) + (alive until b but born before a)

Prop. For $t \leq t' \in (a, b)$, $M(t \longrightarrow t')$ induces $C_I(t) \xrightarrow{\simeq} C_I(t')$

Def. $C_I(M) := \varprojlim_{t \in I} C_I(t)$

Ideas of proof : interval-decomposition
(Crawley-Boevey)

1. Define a *counting functor* for each interval I :

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3. Show that $M = \bigoplus_I M_I$
   - show that the M_I's are in direct sum
   - show that the sum of the M_I's covers M

Idea of proof : rectangle-decomposition

1. Define a *counting functor* for each **rectangle** R :

$$C_R : \textbf{w.exact } \mathrm{vec}_{\mathbf{k}}^{\mathbb{R}} \to \mathrm{vec}_{\mathbf{k}}$$

$$M \mapsto \mathbf{k}^{\mathrm{mult}(\mathbf{k}_R; M)} \qquad \text{where } \mathrm{mult}(\mathbf{k}_R; M) := \max\{n \mid M \simeq \mathbf{k}_R^n \oplus N\}$$

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Idea of proof : rectangle-decomposition

1. Define a *counting functor* for each **rectangle** R :

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Pb : product order on $\mathbb{R}^2$ is not total

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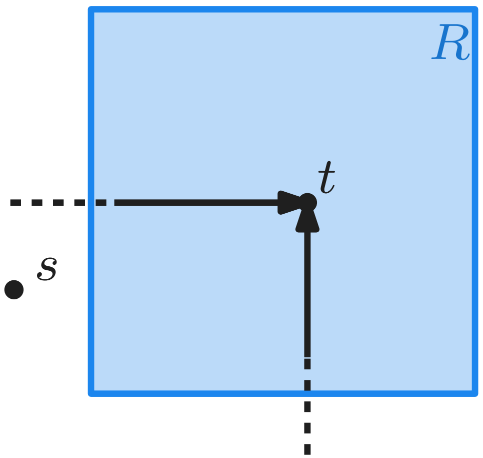

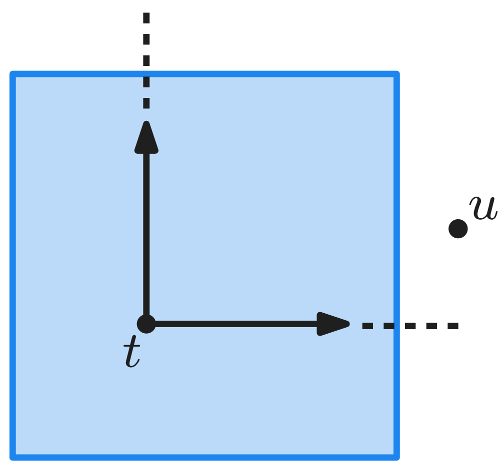

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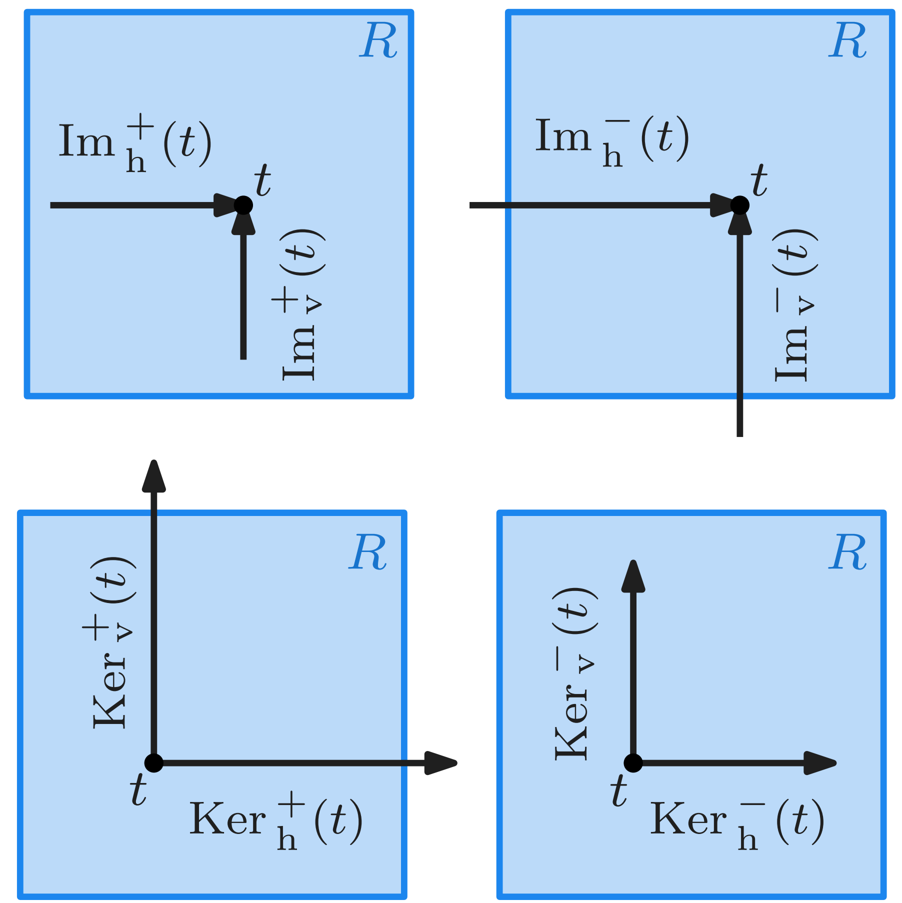

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Then : Define $C_R(M)$ similarly

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Summary

(i) Interval decomposability is not local

(ii) Block & rectangle decomposability are local

(iii) Rectangle decomposability is the biggest local subclass of interval-dec.

Open questions

Partial local characterization ?

- existence of interval summands in the decomposition
- extraction of those interval summands

Classes $\mathcal{I}$ of indecomposables beyond $\mathrm{Int}(P)$ **?**

- local characterization
- compute decomposition

Posets beyond $\mathbb{R}^2$ **?**

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- others ?

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Thank you !

`arxiv:2008.02345`
`arxiv:2002.08894`