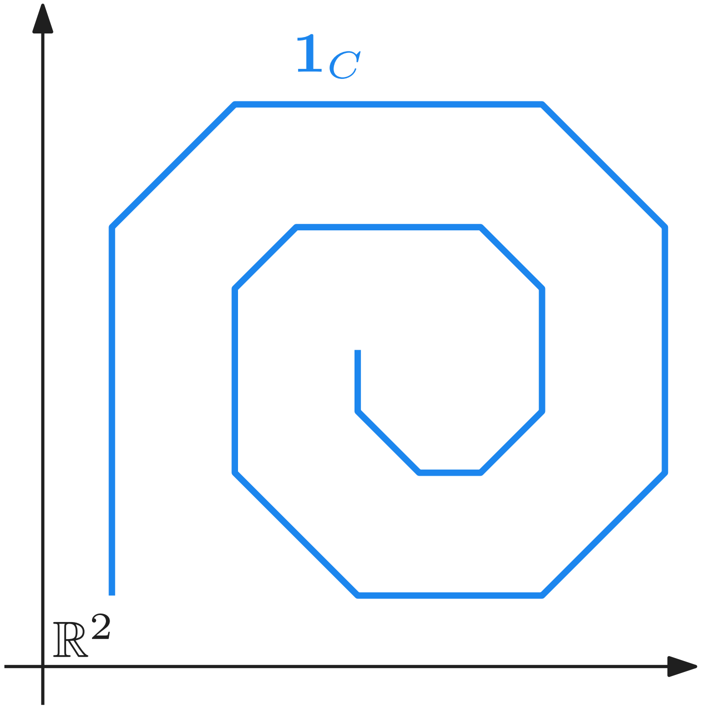
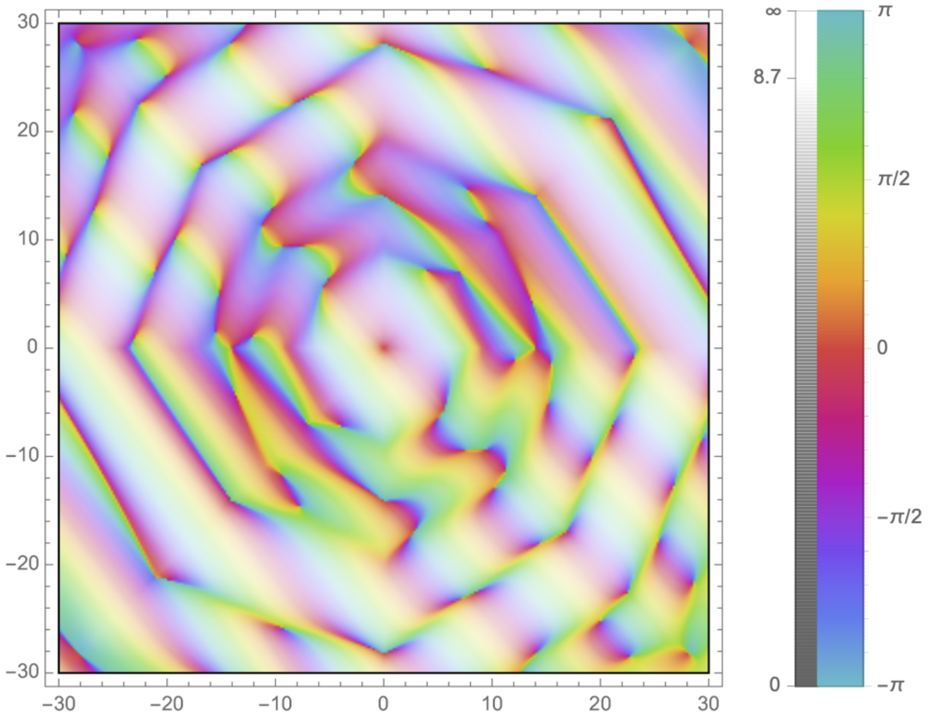
Hybrid transforms of constructible functions

Vadim Lebovici

`arXiv:2111.07829`



$\mapsto$



Applied topology seminar (EPFL) - 22/04/25

Overview

Integral transforms

$$f : \mathbb{R}^n \to \mathbb{C} \quad \text{integrable} \quad \xrightarrow{\text{Fourier}} \quad \underbrace{\mathcal{F}[f] : \mathbb{R}^n \to \mathbb{C}}_{\text{records spectral info.}}$$

$$\varphi : \mathbb{R}^n \to \mathbb{Z} \quad \text{constructible} \quad \xrightarrow{\text{Hybrid Fourier}} \quad \underbrace{\mathcal{E}\mathcal{F}[\varphi] : \mathbb{R}^n \to \mathbb{C}}_{\text{records topological info.}}$$

Overview

Integral transforms

$$f : \mathbb{R}^n \to \mathbb{C} \quad \text{integrable} \quad \xrightarrow{\text{Fourier}} \quad \underbrace{\mathcal{F}[f] : \mathbb{R}^n \to \mathbb{C}}_{\text{records spectral info.}}$$

$$\varphi : \mathbb{R}^n \to \mathbb{Z} \quad \text{constructible} \quad \xrightarrow{\text{Hybrid Fourier}} \quad \underbrace{\mathcal{E}\mathcal{F}[\varphi] : \mathbb{R}^n \to \mathbb{C}}_{\text{records topological info.}}$$

Outcomes of hybrid transforms

- Informative
- Adapted to statistical tools
- Efficiently computable
- Well-behaved (invariance, regularity, mean formulae)
- Generalize existing invariants (e.g. Persistent magnitude [8],
  Euler characteristic of barcodes [9])

[8] Govc, Hepworth (2021) *Persistent magnitude*. Journal of Pure and Applied Algebra, 225(3), 106517.

[9] Bobrowski, Borman (2012) *Euler integration of Gaussian random fields and persistent homology*. Journal of Topology and Analysis, 4(01), 49-70.

Overview

Integral transforms

$$f : \mathbb{R}^n \to \mathbb{C} \quad \text{integrable} \xrightarrow{\text{Fourier}} \underbrace{\mathcal{F}[f] : \mathbb{R}^n \to \mathbb{C}}_{\text{records spectral info.}}$$

$$\varphi : \mathbb{R}^n \to \mathbb{Z} \quad \underline{\text{constructible}} \xrightarrow{\text{Hybrid Fourier}} \underbrace{\mathcal{E}\mathcal{F}[\varphi] : \mathbb{R}^n \to \mathbb{C}}_{\text{records topological info.}}$$

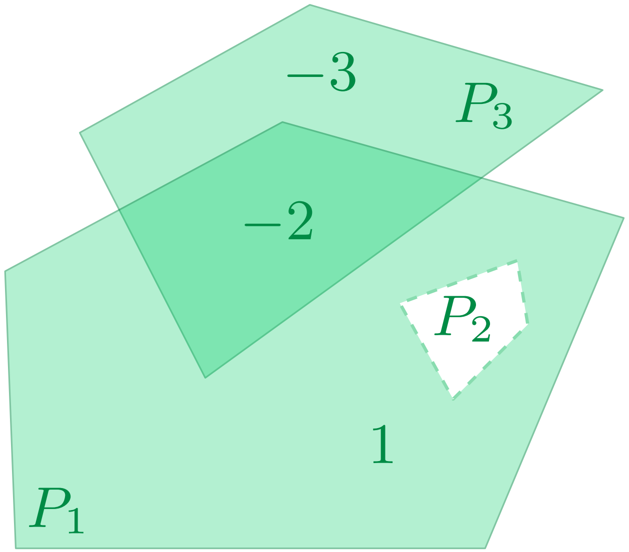
Def.

$$\varphi = \sum_{i=1}^{k} m_i \mathbf{1}_{K_i}$$

where
- $m_i \in \mathbb{Z}$
- K_i compact subanalytic in $\mathbb{R}^n$ (e.g. polytope)

Not. $\varphi \in \mathrm{CF}(\mathbb{R}^n)$

Ex.



$$\varphi = \mathbf{1}_{P_1} - \mathbf{1}_{P_2} - 3 \cdot \mathbf{1}_{P_3} \in \mathrm{CF}_{\mathrm{PL}}(\mathbb{R}^2)$$

Overview

Integral transforms

$f : \mathbb{R}^n \to \mathbb{C}$ integrable $\xrightarrow{\text{Fourier}}$ $\underbrace{\mathcal{F}[f] : \mathbb{R}^n \to \mathbb{C}}_{\text{records spectral info.}}$

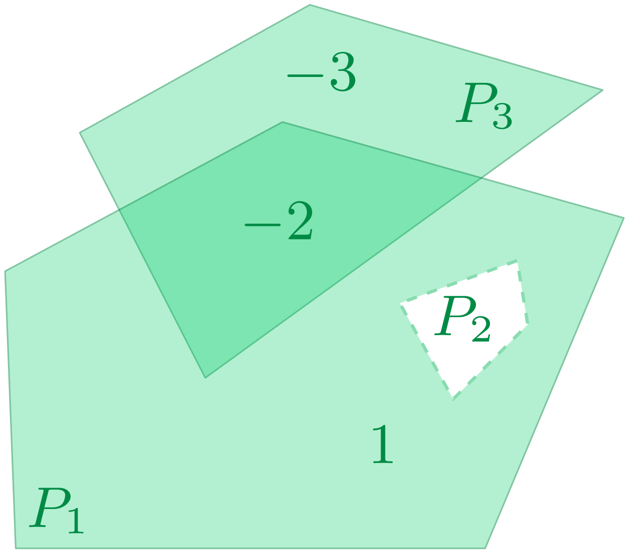
$\varphi : \mathbb{R}^n \to \mathbb{Z}$ $\underline{\text{constructible}}$ $\xrightarrow{\text{Hybrid Fourier}}$ $\underbrace{\mathcal{E}\mathcal{F}[\varphi] : \mathbb{R}^n \to \mathbb{C}}_{\text{records topological info.}}$

Def.

$$\varphi = \sum_{i=1}^{k} m_i \mathbf{1}_{K_i}$$

where
- $m_i \in \mathbb{Z}$
- K_i compact subanalytic in $\mathbb{R}^n$ (e.g. polytope)

Ex.



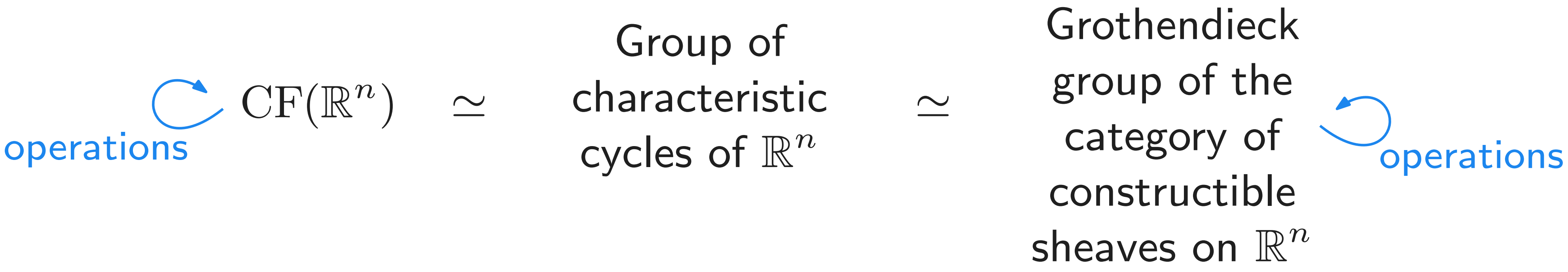
$$\varphi = \mathbf{1}_{P_1} - \mathbf{1}_{P_2} - 3 \cdot \mathbf{1}_{P_3} \in \mathrm{CF}_{\underline{\mathrm{PL}}}(\mathbb{R}^2)$$

Not. $\varphi \in \mathrm{CF}(\mathbb{R}^n)$

Why constructible functions ?

Why constructible functions?

1. Theoretically rich [5]



operations

$\mathrm{CF}(\mathbb{R}^n) \simeq$

Group of characteristic cycles of $\mathbb{R}^n$ $\simeq$

Grothendieck group of the category of constructible sheaves on $\mathbb{R}^n$

operations

---

[5] Kashiwara, Schapira (1990) *Sheaves on Manifolds* (Vol. 292). Springer Science & Business Media.

Why constructible functions ?

1. Theoretically rich

2. Useful in applied topology

   (i) Persistence

$$\underbrace{M = \bigoplus_{j\in\mathbb{Z}} \overset{\text{pers. mod. on } \mathbb{R}^n}{M_j}}_{\text{graded pers. mod. on } \mathbb{R}^n} \longmapsto \overbrace{\varphi_M : x \in \mathbb{R}^n \mapsto \sum_{j\in\mathbb{Z}} (-1)^j \dim M_j(x)}^{\text{invariant of pers. mod.}}$$

Why constructible functions ?

1. Theoretically rich

2. Useful in applied topology

   (i) Persistence

$$M = \bigoplus_{j\in\mathbb{Z}} \overset{\text{pers. mod. on } \mathbb{R}^n}{M_j} \longmapsto \overbrace{\varphi_M : x \in \mathbb{R}^n \mapsto \sum_{j\in\mathbb{Z}} (-1)^j \dim M_j(x)}^{\text{invariant of pers. mod.}}$$

graded pers. mod. on $\mathbb{R}^n$

Ex. $(n = 1)$ For $f : X \to \mathbb{R}$ continuous subanalytic,

$$\varphi_f : \begin{array}{l} \mathbb{R} \to \mathbb{Z} \\ t \mapsto \chi\big(\{x \in X \ ; \ f(x) \leq t\}\big) \end{array} \qquad \in \mathrm{CF}(\mathbb{R})$$

Here $M = \bigoplus_{j\in\mathbb{Z}} \mathrm{PH}_j(X, f)$

Why constructible functions ?

1. Theoretically rich

2. Useful in applied topology

   (i) Persistence

pers. mod. on $\mathbb{R}^n$

invariant of pers. mod.

$$M = \bigoplus_{j\in\mathbb{Z}} M_j \longmapsto \overbrace{\varphi_M : x \in \mathbb{R}^n \mapsto \sum_{j\in\mathbb{Z}} (-1)^j \dim M_j(x)}$$

graded pers. mod. on $\mathbb{R}^n$

Ex. $(n = 1)$ For $f : X \to \mathbb{R}$ continuous subanalytic,

$$\varphi_f : \begin{array}{l} \mathbb{R} \to \mathbb{Z} \\ t \mapsto \underbrace{\chi\big(\{x \in X \ ; \ f(x) \leq t\}\big)} \end{array} \in \mathrm{CF}(\mathbb{R})$$

Def. Euler characteristic

$$\chi(K) = \sum_{j\in\mathbb{Z}} (-1)^j \dim H_j(K;\mathbb{Q})$$

compact subanalytic

$\chi(K) = \sum_{j\in\mathbb{Z}} (-1)^j \#\{j\text{-simplices}\}$ if K simp. cplx

$\chi(K) = 1$ if K compact convex

Why constructible functions ?

1. Theoretically rich

2. Useful in applied topology

   (i) Persistence

pers. mod. on $\mathbb{R}^n$

invariant of pers. mod.

$$M = \bigoplus_{j\in\mathbb{Z}} M_j \longmapsto \overbrace{\varphi_M : x \in \mathbb{R}^n \mapsto \sum_{j\in\mathbb{Z}} (-1)^j \dim M_j(x)}$$

graded pers. mod. on $\mathbb{R}^n$

Ex. $(n = 1)$ For $f : X \to \mathbb{R}$ continuous subanalytic,

Euler curve →

$$\left[\varphi_f : \begin{array}{l} \mathbb{R} \to \mathbb{Z} \\ t \mapsto \chi\big(\{x \in X \ ; \ f(x) \leq t\}\big) \end{array}\right. \qquad \in \mathrm{CF}(\mathbb{R})$$

- Faster to compute
- Generalizes to $n \geq 1$ (multi-parameter persistance)

Ex. $(n = 2)$ Euler surfaces [10] → detection of diabetic retinopathy

[10] Beltramo, Andreeva, Giarratano, Bernabeu, Sarkar, Skraba (2021) *Euler Characteristic Surfaces* arXiv preprint :2102.08260

Why constructible functions ?

1. Theoretically rich
2. Useful in applied topology
    - (i) Persistence

pers. mod. on $\mathbb{R}^n$

$$M = \underbrace{\bigoplus_{j\in\mathbb{Z}} M_j}_{\text{graded pers. mod. on } \mathbb{R}^n} \longmapsto \overbrace{\varphi_M : x \in \mathbb{R}^n \mapsto \sum_{j\in\mathbb{Z}} (-1)^j \dim M_j(x)}^{\text{invariant of pers. mod.}}$$

Ex. $(n = 1)$ For $f : X \to \mathbb{R}$ continuous subanalytic,

Euler curve →

$$\varphi_f : \left[\begin{array}{l} \mathbb{R} \to \mathbb{Z} \\ t \mapsto \chi\big(\{x \in X \; ; \; f(x) \leq t\}\big) \end{array} \right. \in \mathrm{CF}(\mathbb{R})$$

Too rough summary ?

- Faster to compute
- Generalizes to $n \geq 1$ (multi-parameter persistance)

Ex. $(n = 2)$ Euler surfaces [10] → detection of diabetic retinopathy

[10] Beltramo, Andreeva, Giarratano, Bernabeu, Sarkar, Skraba (2021) *Euler Characteristic Surfaces* arXiv preprint :2102.08260

Why constructible functions ?

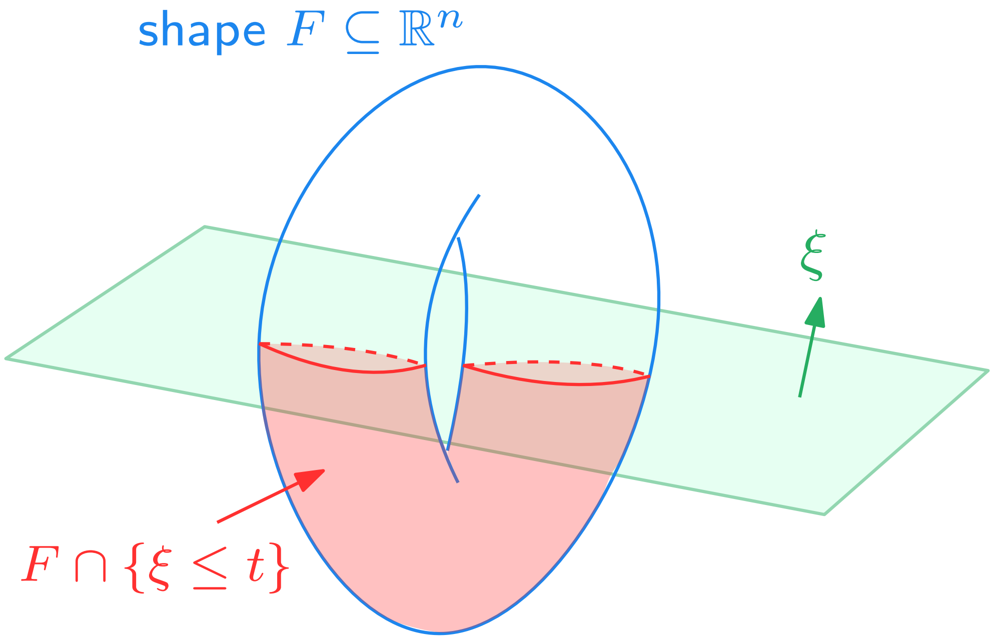
1. Theoretically rich

2. Useful in applied topology

   (i) Persistence

   (ii) Euler characteristic transform

[7] (Curry, Mukherjee, Turner 2018)
[11] (Turner, Mukherjee, Boyer 2014)
[12] (Ghrist, Levanger, Mai 2018)



$$\varphi_\xi : t \mapsto \chi(F \cap \{\xi \leq t\})$$

Que. Can one recover F from Euler curves for all height functions ?

Why constructible functions ?

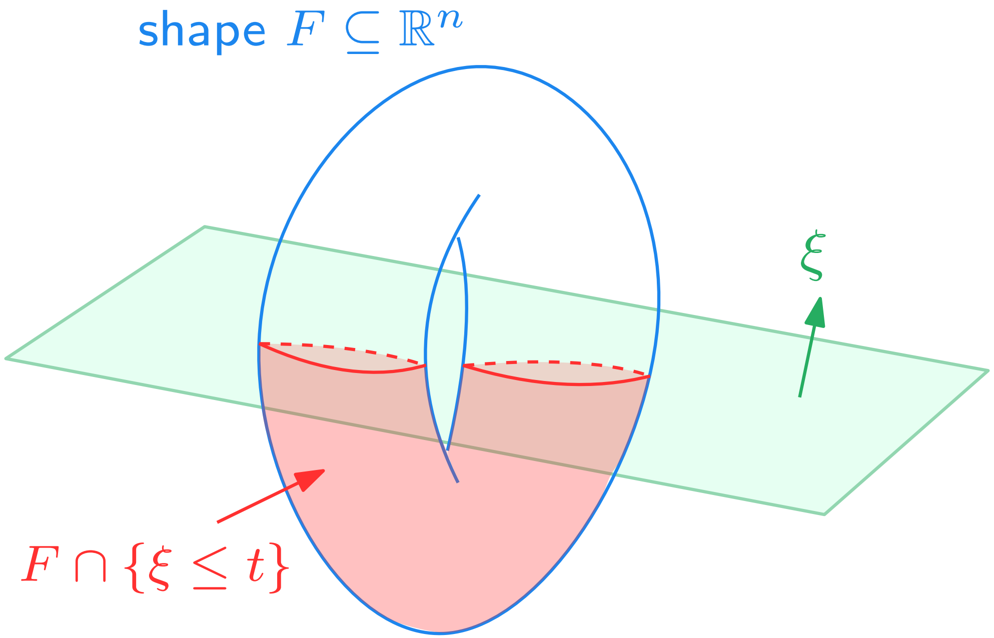
1. Theoretically rich

2. Useful in applied topology

   (i) Persistence

   (ii) Euler characteristic transform

[7] (Curry, Mukherjee, Turner 2018)
[11] (Turner, Mukherjee, Boyer 2014)
[12] (Ghrist, Levanger, Mai 2018)



$$\varphi_\xi : t \mapsto \chi(F \cap \{\xi \leq t\})$$

Que. Can one recover F from Euler curves for all height functions ?

Thm. The map $\xi \mapsto \varphi_\xi$ fully determines F.

Proof. Inversion of the (constructible) Radon transform [4] (Schapira 1995)

$$\mathcal{R} : \mathrm{CF}(\mathbb{R}^3) \to \mathrm{CF}(\mathbb{P}_3^*)$$

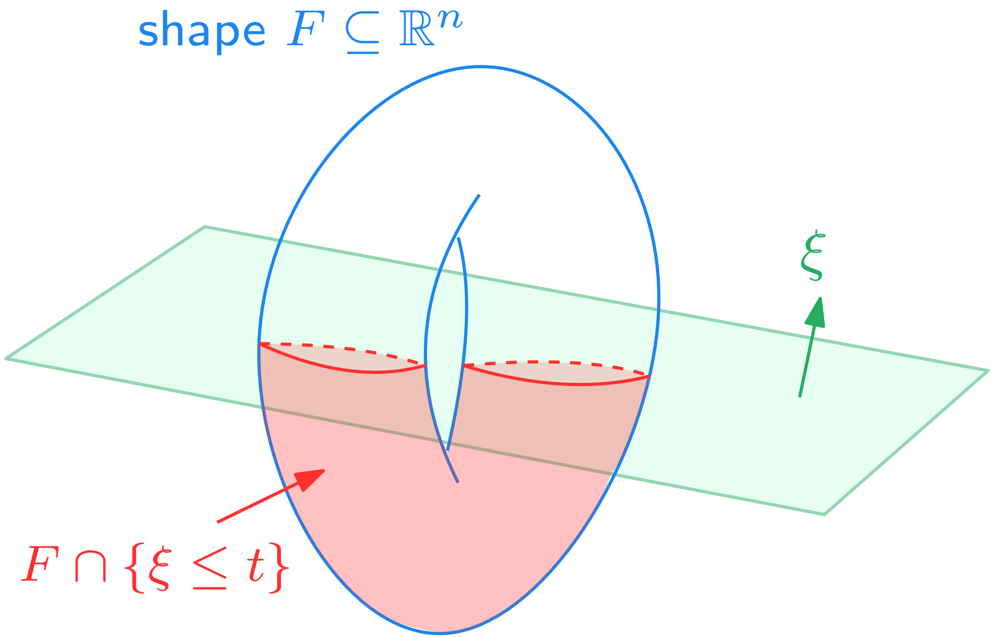
Why constructible functions ?

1. Theoretically rich

2. Useful in applied topology

   (i) Persistence

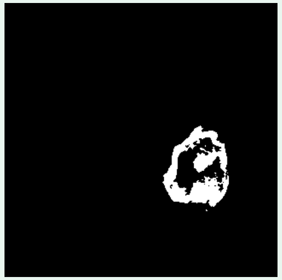
   (ii) Euler characteristic transform



$$\varphi_\xi : t \mapsto \chi(F \cap \{\xi \leq t\})$$

Ex. [1]

Prediction of clinical outcomes in brain tumors



[1] Crawford, Monod, Chen, Mukherjee, Rabadán (2020) *Predicting Clinical Outcomes in Glioblastoma : An Application of Topological and Functional Data Analysis*, Journal of the American Statistical Association, 115 :531, 1139-1150

Why constructible functions ?

1. Theoretically rich

2. Useful in applied topology

   (i) Persistence

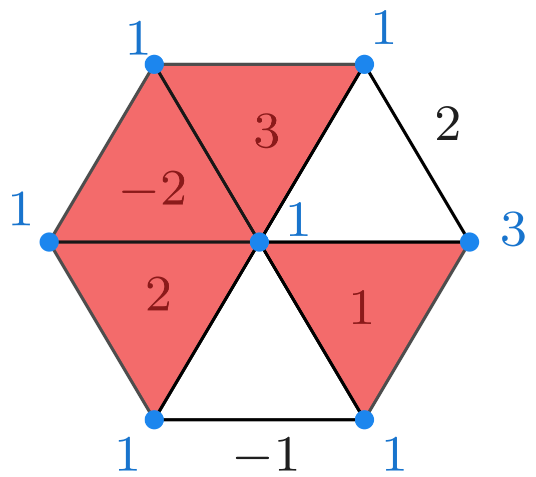
   (ii) Euler characteristic transform

3. Ubiquitous in computer science (implementable by nature)

e.g. (weighted) simplicial complex

Ex. [13] Biomolecular data analysis

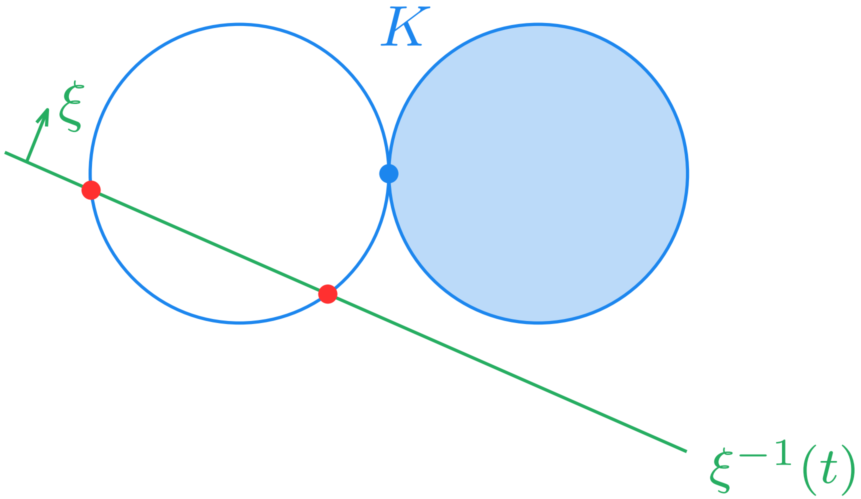
[13] Meng, Anand, Lu, Wu, Xia (2020) *Weighted persistent homology for biomolecular data analysis*. Scientific reports, 10(1), 1-15.



An operation : the pushforward

Consider $\xi : \mathbb{R}^n \to \mathbb{R}$ linear.

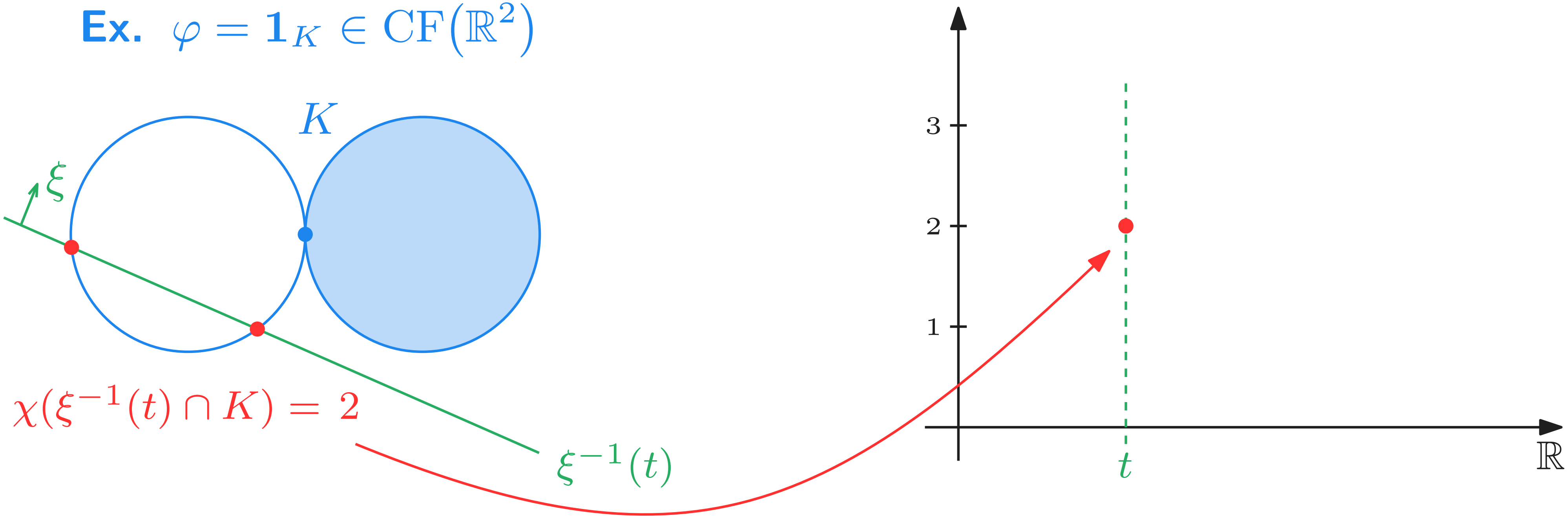
Ex. $\varphi = \mathbf{1}_K \in \mathrm{CF}(\mathbb{R}^2)$



An operation : the pushforward

Consider $\xi : \mathbb{R}^n \to \mathbb{R}$ linear.

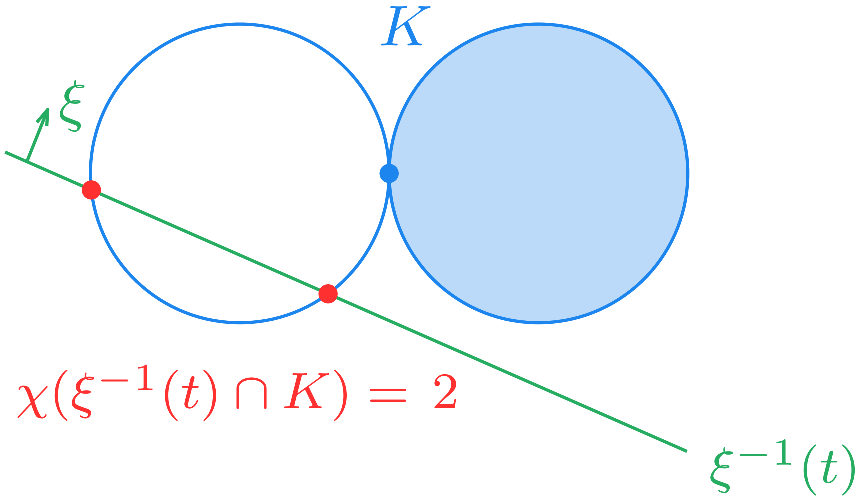
Ex. $\varphi = \mathbf{1}_K \in \mathrm{CF}(\mathbb{R}^2)$



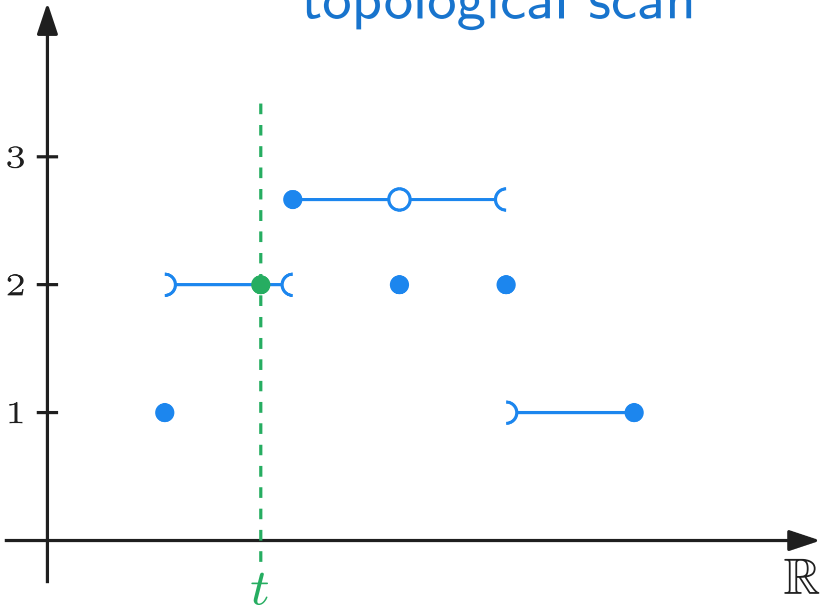
An operation : the pushforward

Consider $\xi : \mathbb{R}^n \to \mathbb{R}$ linear.

Ex. $\varphi = \mathbf{1}_K \in \mathrm{CF}(\mathbb{R}^2)$



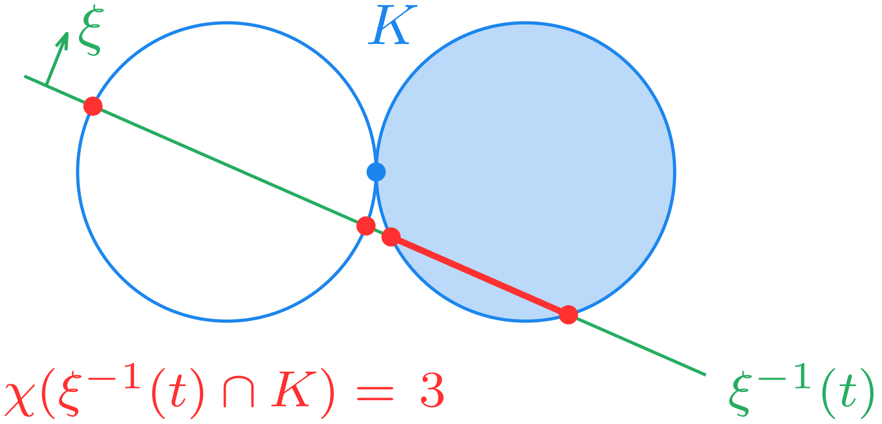
"topological scan"

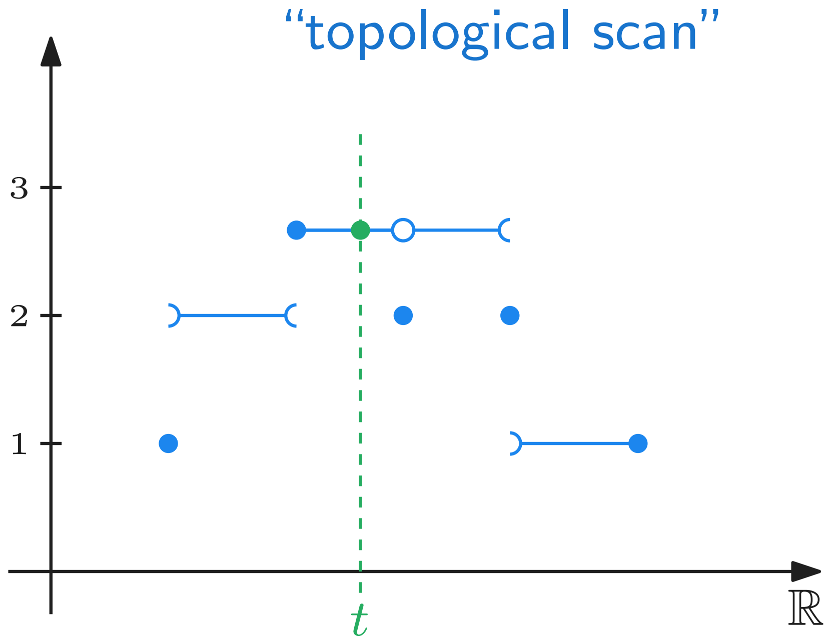


An operation : the pushforward

Consider $\xi : \mathbb{R}^n \to \mathbb{R}$ linear.

Ex. $\varphi = \mathbf{1}_K \in \mathrm{CF}(\mathbb{R}^2)$

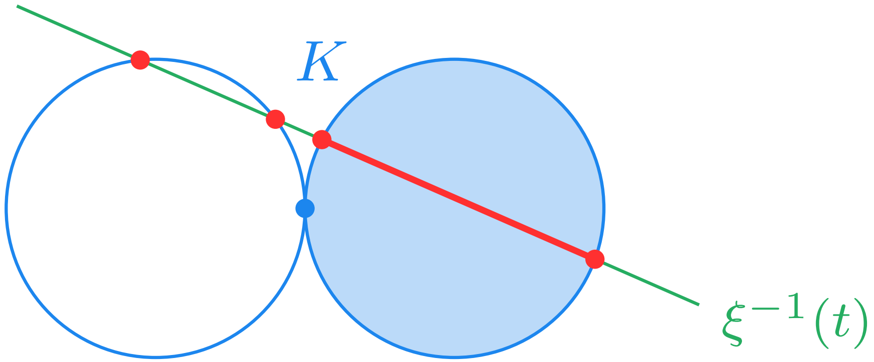




An operation : the pushforward

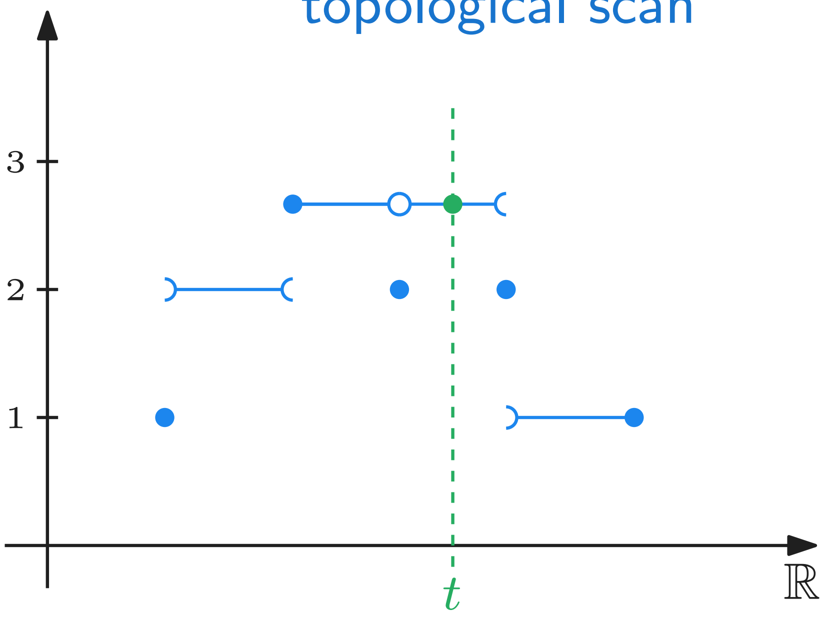
Consider $\xi : \mathbb{R}^n \to \mathbb{R}$ linear.

Ex. $\varphi = \mathbf{1}_K \in \mathrm{CF}(\mathbb{R}^2)$



$\chi(\xi^{-1}(t) \cap K) = 3$

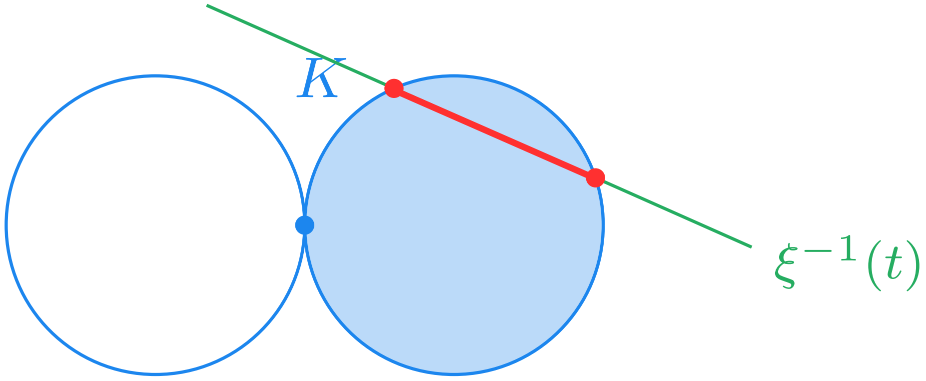
"topological scan"



An operation : the pushforward

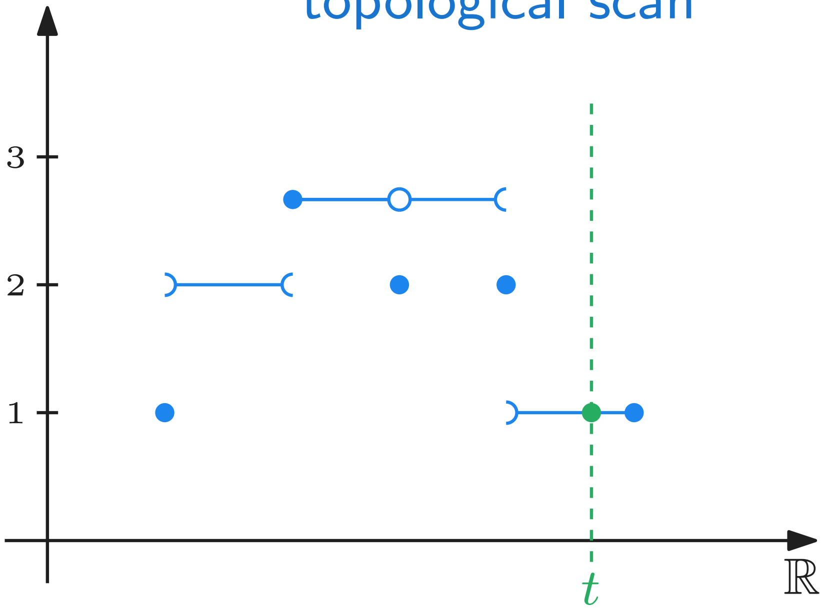
Consider $\xi : \mathbb{R}^n \to \mathbb{R}$ linear.

Ex. $\varphi = \mathbf{1}_K \in \mathrm{CF}\big(\mathbb{R}^2\big)$



$\chi(\xi^{-1}(t) \cap K) = 1$

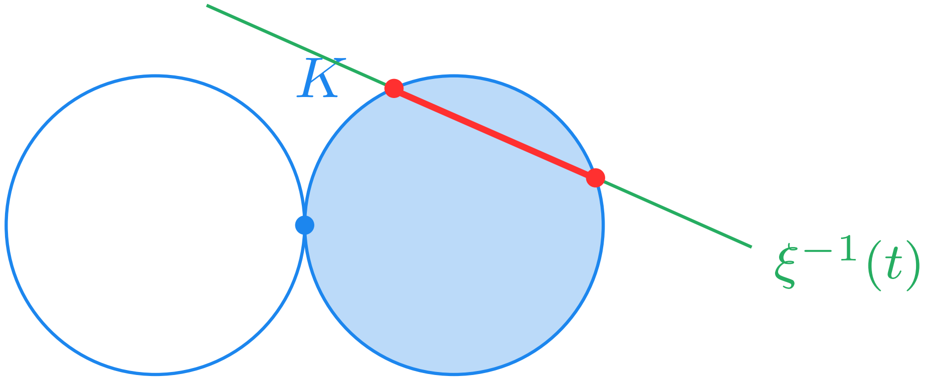
“topological scan”



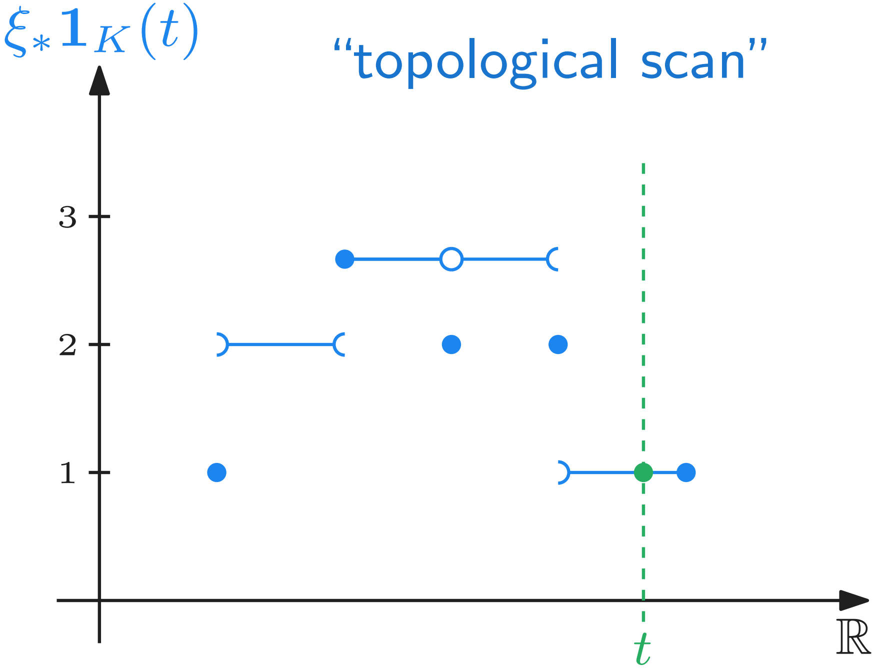
An operation : the pushforward

Consider $\xi : \mathbb{R}^n \to \mathbb{R}$ linear.

Ex. $\varphi = \mathbf{1}_K \in \mathrm{CF}(\mathbb{R}^2)$



$\chi(\xi^{-1}(t) \cap K) = 1$



"topological scan"

Def. (Pushforward) K compact subanalytic

$$\xi_* \mathbf{1}_K : \begin{array}{rcl} \mathbb{R} & \longrightarrow & \mathbb{Z} \\ t & \longmapsto & \chi\left(\xi^{-1}(t) \cap K\right) \end{array} \in \mathrm{CF}(\mathbb{R})$$

(General case)

For $\varphi = \sum m_i \cdot \mathbf{1}_{K_i}$,

$$\xi_* \varphi := \sum m_i \cdot \xi_* \mathbf{1}_{K_i}$$

$\in \mathrm{CF}(\mathbb{R})$

Rk. Topological dimensionality reduction from n to 1 $\quad \xi_* : \mathrm{CF}(\mathbb{R}^n) \to \mathrm{CF}(\mathbb{R})$

Constructible world [6]

---

[6] Curry, Ghrist, Robinson (2012) *Euler calculus with applications to signals and sensing.* Proceedings of Symposia in Applied Mathematics. Vol. 70.

Constructible world

Pros

- Topological operations (pushforward, transforms)

- Implementable

Constructible world

Pros

- Topological operations (pushforward, transforms)

- Implementable

Cons : not the best world for stats

Constructible world

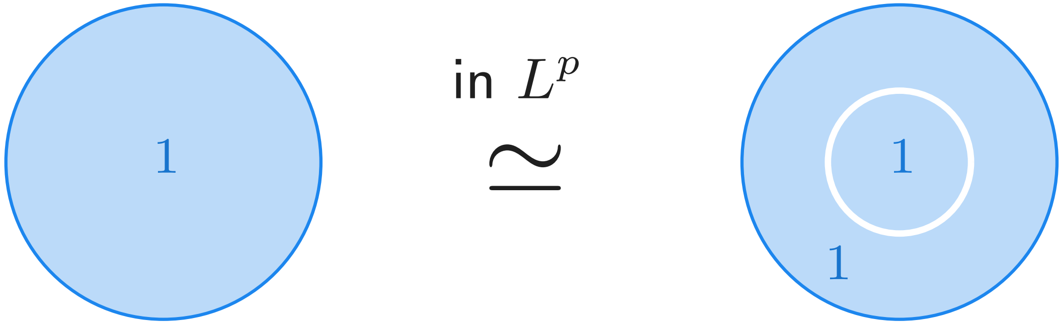
Pros

- Topological operations (pushforward, transforms)

- Implementable

Cons : not the best world for stats

- Topology not accessible with usual stats tools (L^p norms, ...)



Constructible world

Pros

- Topological operations (pushforward, transforms)

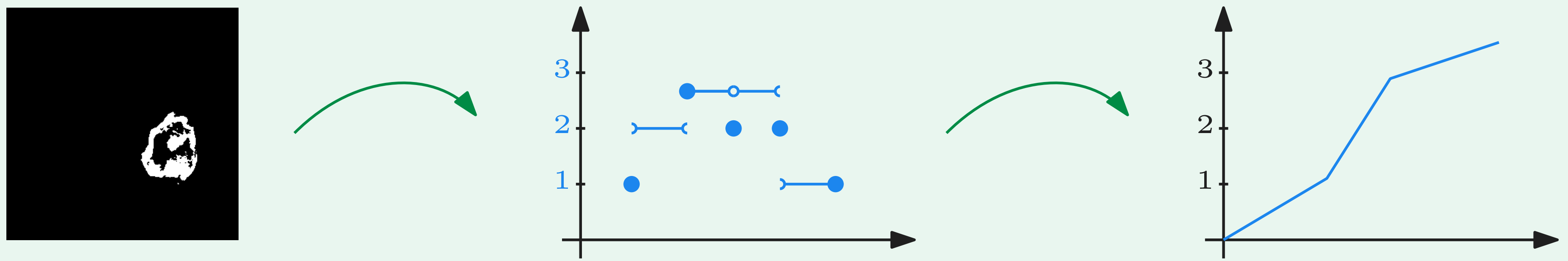
- Implementable

Cons : not the best world for stats

- Topology not accessible with usual stats tools (L^p norms, ...)

- Topological scans do not give access to full scope of functional stats

Ex. [1] Prediction of clinical outcomes in brain tumors



MRI

Euler curves

smoothed Euler curves

---

[1] Crawford, Monod, Chen, Mukherjee, Rabadán (2020) *Predicting Clinical Outcomes in Glioblastoma : An Application of Topological and Functional Data Analysis*, Journal of the American Statistical Association, 115 :531, 1139-1150

Constructible world

Pros

- Topological operations (pushforward, transforms)

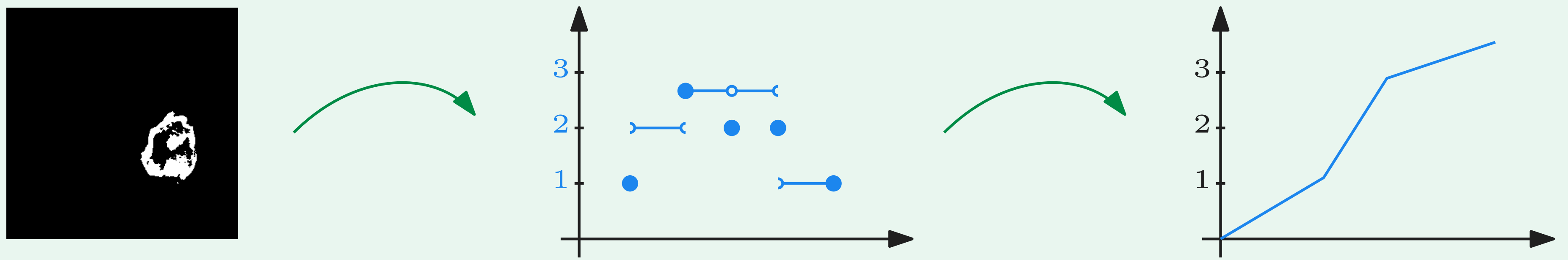
- Implementable

Cons : not the best world for stats

- Topology not accessible with usual stats tools (L^p norms, ...)

- Topological scans do not give access to full scope of functional stats

Ex. [1] Prediction of clinical outcomes in brain tumors



MRI   Euler curves   smoothed Euler curves

Idea. Get out of the constructible world. → **Hybrid transforms**

$$\mathrm{CF}(\mathbb{R}^n) \longrightarrow \mathrm{L}^2(\mathbb{R}^n) \quad \text{(Hilbert space)}$$

[1] Crawford, Monod, Chen, Mukherjee, Rabadán (2020) *Predicting Clinical Outcomes in Glioblastoma : An Application of Topological and Functional Data Analysis*, Journal of the American Statistical Association, 115 :531, 1139-1150

Getting out of the constructible world

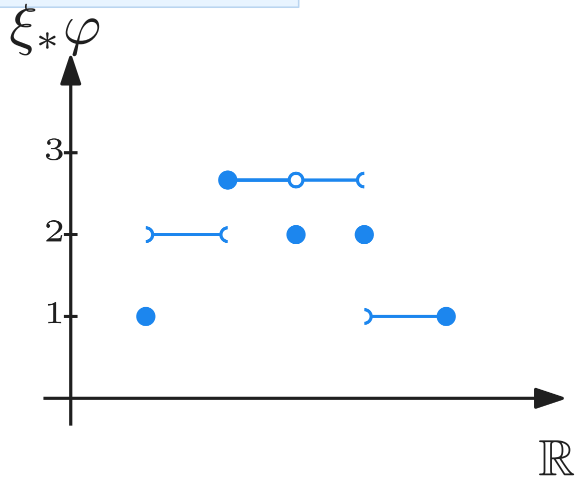
Def. (Hybrid transform) $\kappa : \mathbb{R} \to \mathbb{C}$ in L^1_{loc} and $\varphi \in \mathrm{CF}(\mathbb{R}^n)$

$$\mathrm{T}_\kappa[\varphi] : \begin{array}{rcl} \mathbb{R}^n & \longrightarrow & \mathbb{C} \\ \xi & \longmapsto & \displaystyle\int_{\mathbb{R}} \kappa(t)\xi_*\varphi(t)\,\mathrm{d}t \end{array}$$

Getting out of the constructible world

Def. (Hybrid transform) $\kappa : \mathbb{R} \to \mathbb{C}$ in L^1_{loc} and $\varphi \in \mathrm{CF}(\mathbb{R}^n)$

$$\mathrm{T}_\kappa[\varphi] : \begin{array}{rcl} \mathbb{R}^n & \longrightarrow & \mathbb{C} \\ \xi & \longmapsto & \overbrace{\int_{\mathbb{R}} \kappa(t)\underbrace{\xi_*\varphi(t)}_{\text{topological dim. reduction}}\,\mathrm{d}t}^{\text{integration against kernel}} \end{array}$$

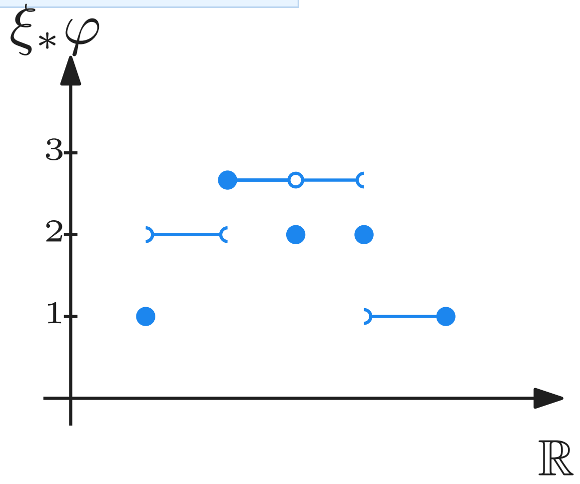


Getting out of the constructible world

Def. (Hybrid transform) $\kappa : \mathbb{R} \to \mathbb{C}$ in L^1_{loc} and $\varphi \in \mathrm{CF}(\mathbb{R}^n)$

$$\mathrm{T}_\kappa[\varphi] : \begin{array}{rcl} \mathbb{R}^n & \longrightarrow & \mathbb{C} \\ \xi & \longmapsto & \overbrace{\int_{\mathbb{R}} \kappa(t) \underbrace{\xi_*\varphi(t)}_{\text{topological dim. reduction}} \, \mathrm{d}t}^{\text{integration against kernel}} \end{array}$$

First appear in [3] (without kernel κ)



Ex. Euler-Fourier

$$\mathcal{EF}[\varphi] : \begin{array}{rcl} \mathbb{R}^n & \longrightarrow & \mathbb{C} \\ \xi & \longmapsto & \int_{\mathbb{R}} e^{-it} \xi_*\varphi(t) \, \mathrm{d}t \end{array}$$

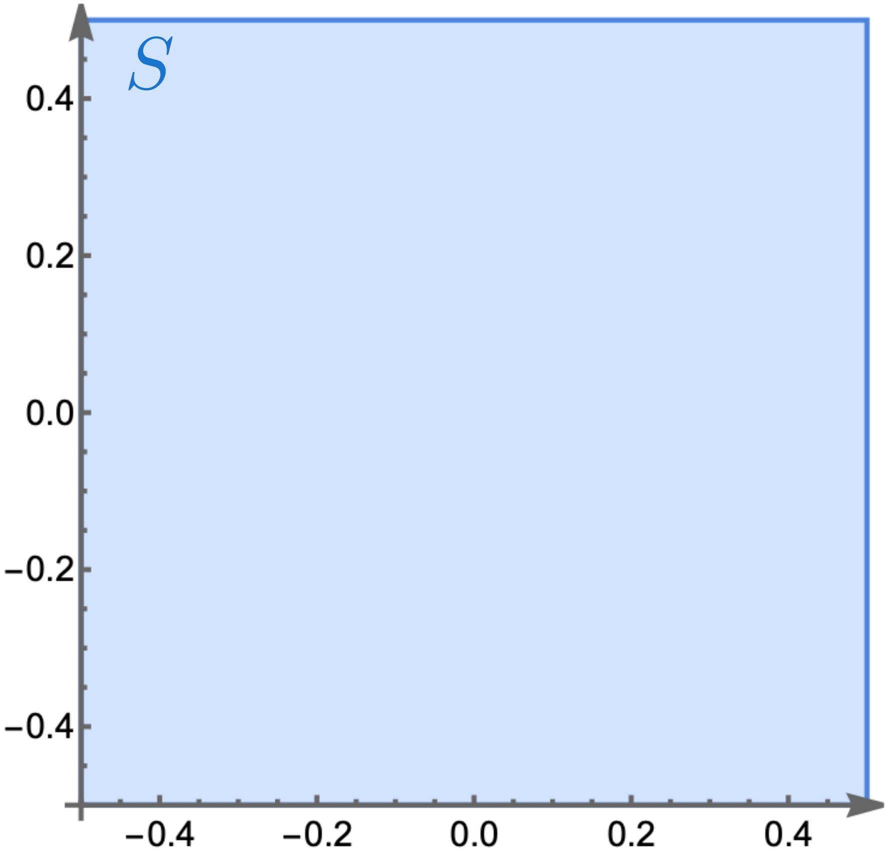
Ex. Euler-Laplace

$$\mathcal{EL}[\varphi] : \begin{array}{rcl} \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ \xi & \longmapsto & \int_{\mathbb{R}} e^{-t} \xi_*\varphi(t) \, \mathrm{d}t \end{array}$$

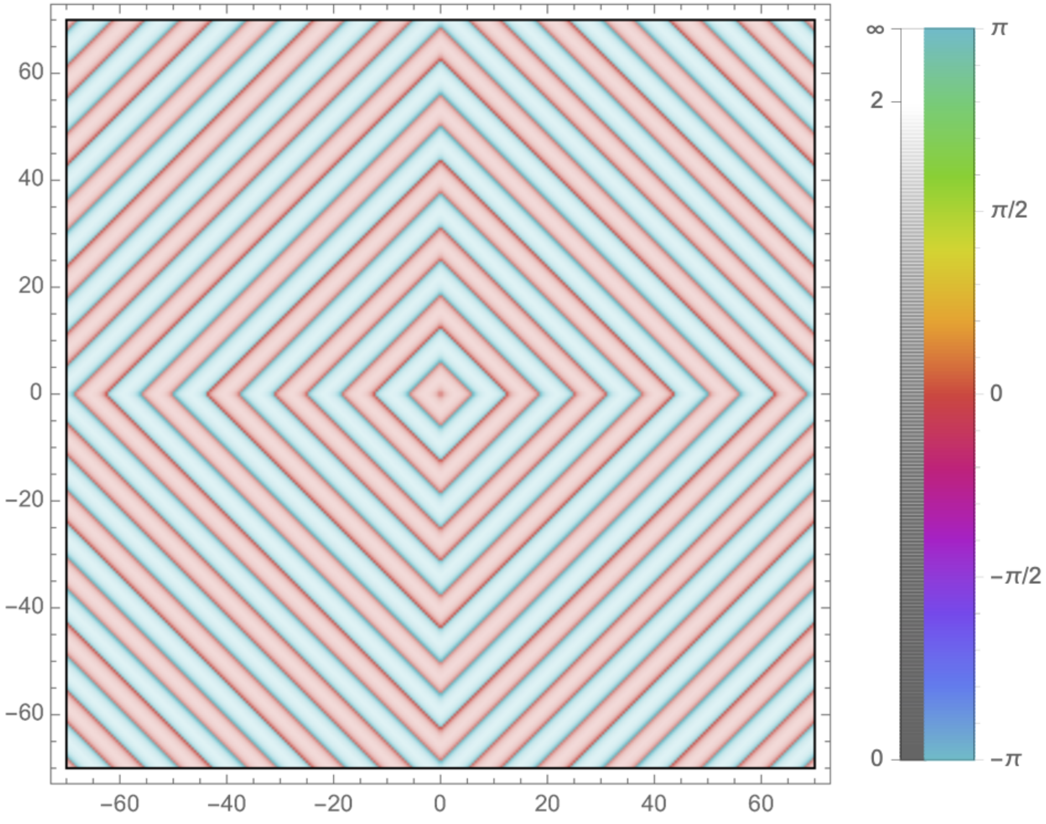
Multi-parameter persistent magnitude

[3] Ghrist, Robinson (2011) *Euler–Bessel and Euler–Fourier transforms*. Inverse problems, 27(12), 124006.

Toy example : square

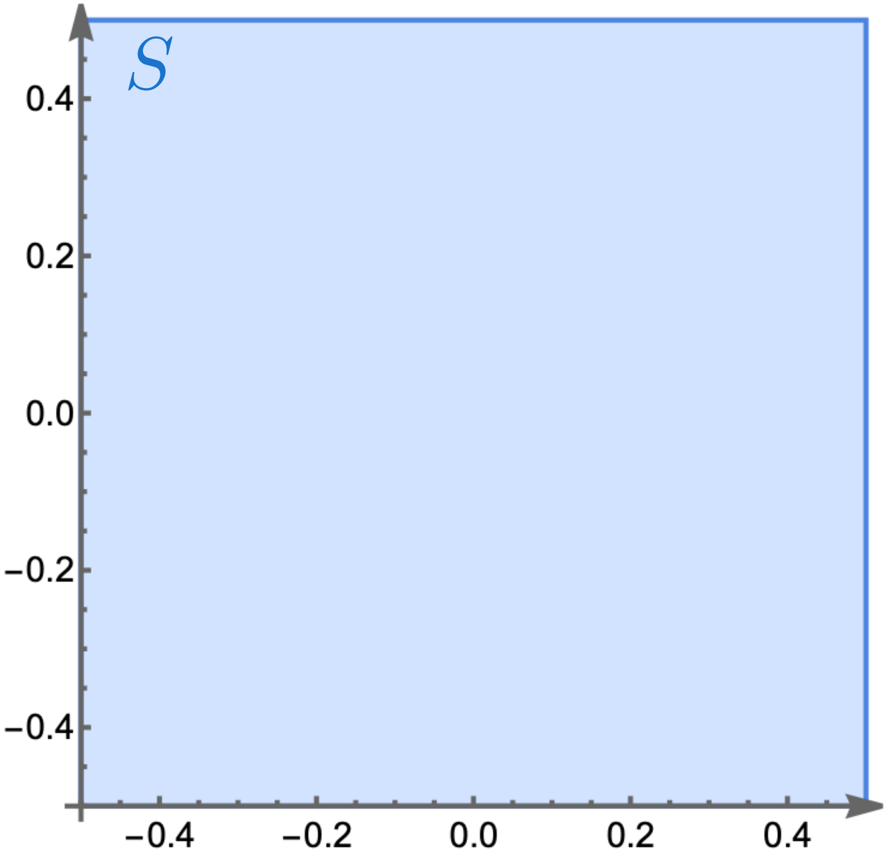


$$\mathbf{1}_S \in \mathrm{CF}\big(\mathbb{R}^2\big)$$

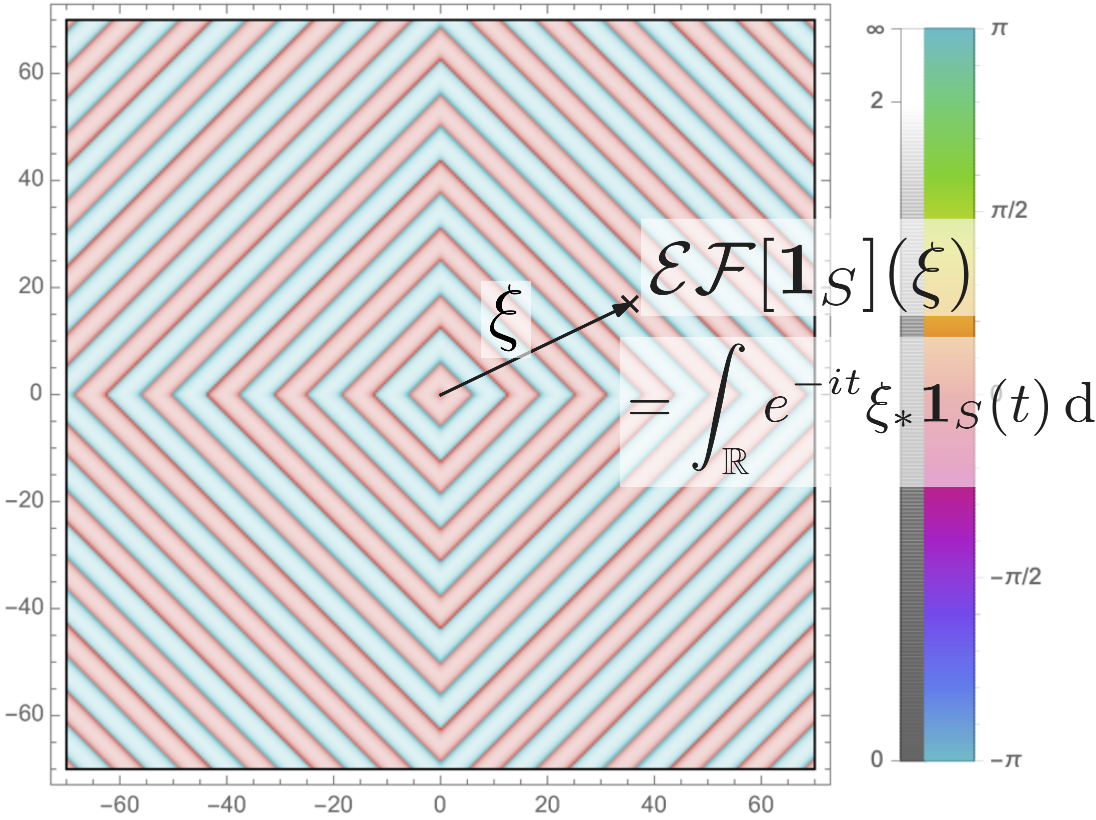


$$\mathcal{E}\mathcal{F}[\mathbf{1}_S] : \mathbb{R}^2 \to \mathbb{C}$$

Toy example : square



$$\mathbf{1}_S \in \mathrm{CF}\big(\mathbb{R}^2\big)$$

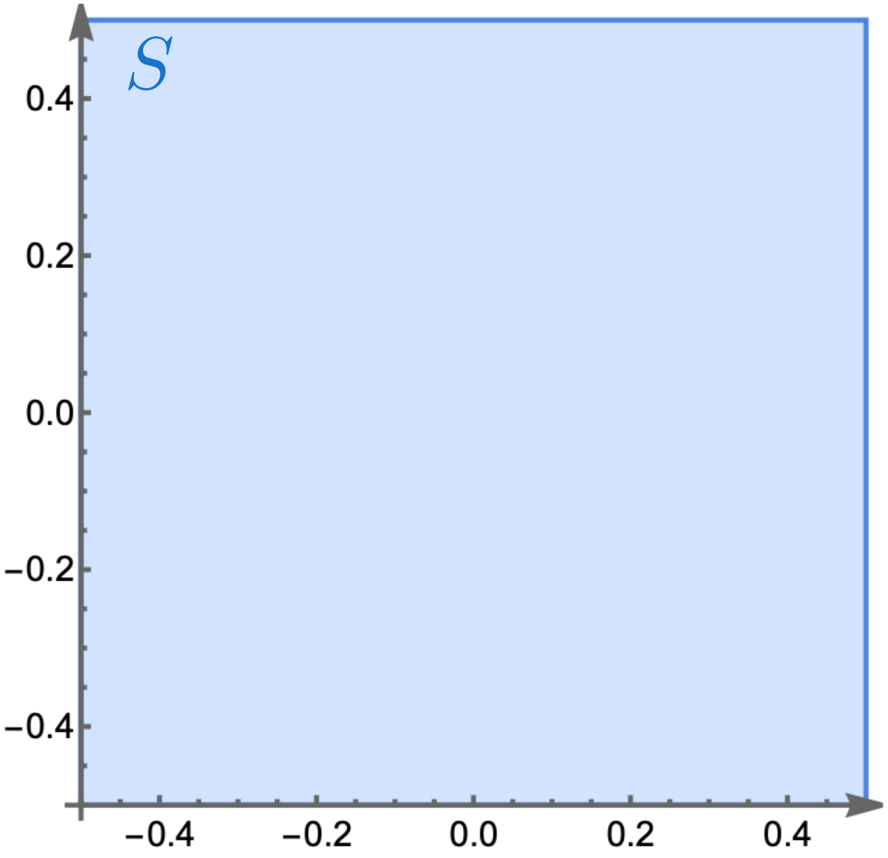


$$\mathcal{E}\mathcal{F}[\mathbf{1}_S](\xi)$$

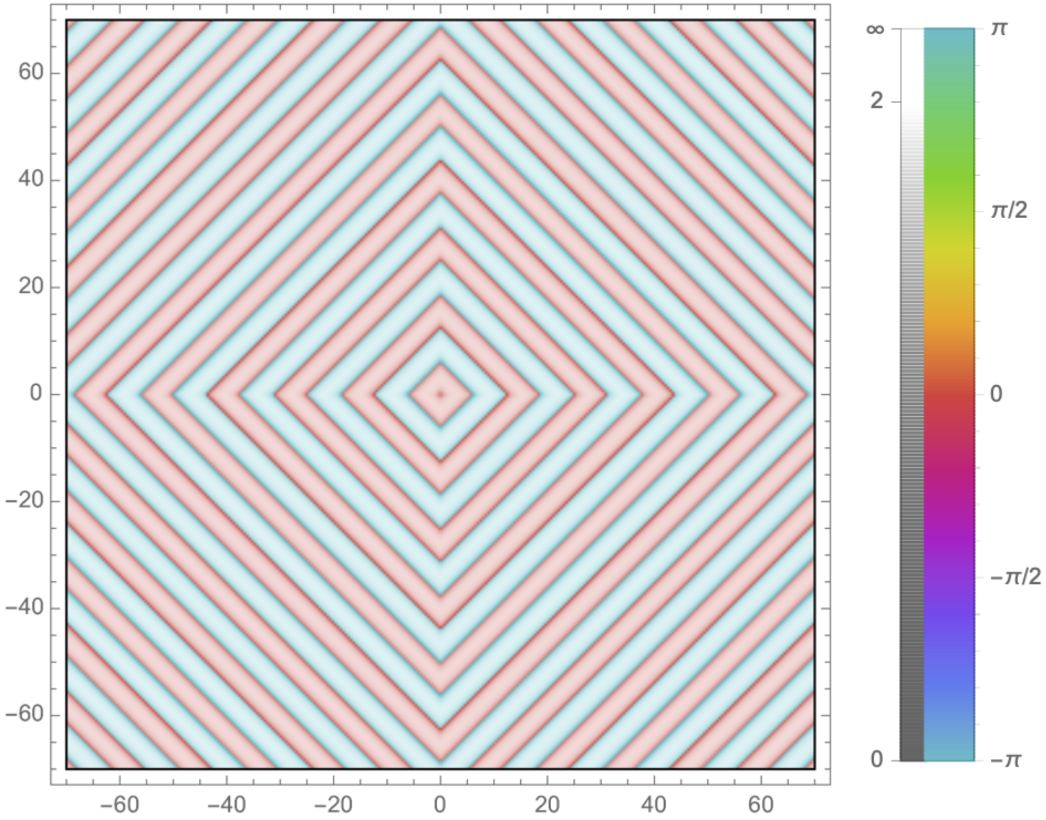
$$= \int_{\mathbb{R}} e^{-it}\xi_* \mathbf{1}_S(t)\,\mathrm{d}$$

$$\mathcal{E}\mathcal{F}[\mathbf{1}_S] : \mathbb{R}^2 \to \mathbb{C}$$

Toy example : square

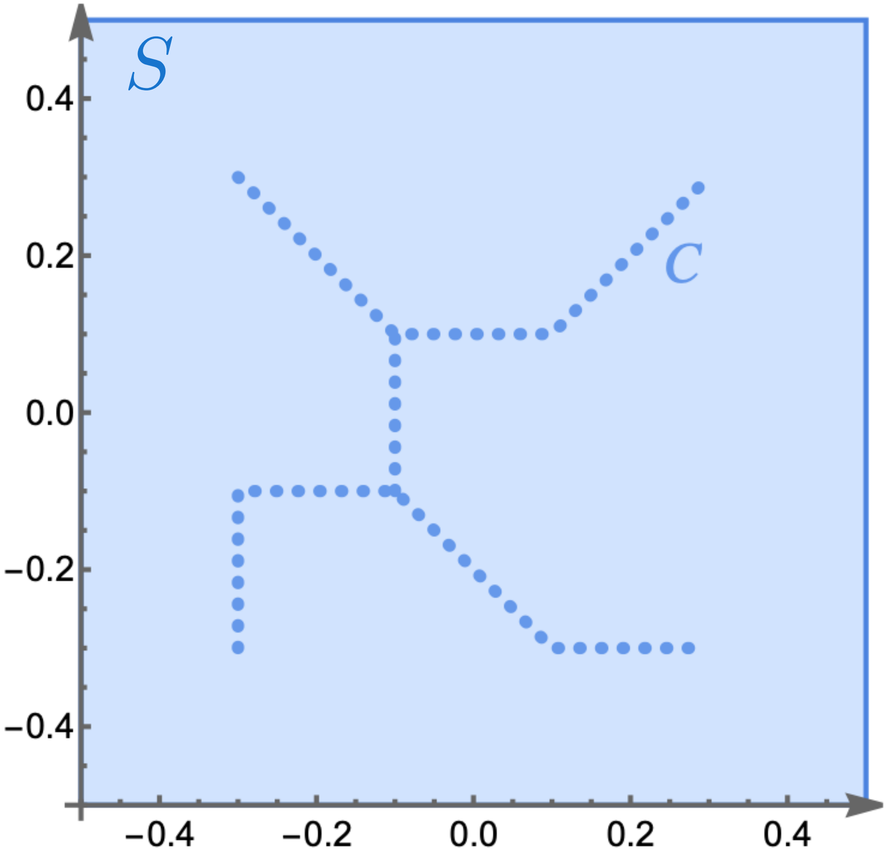


$$\mathbf{1}_S \in \mathrm{CF}\big(\mathbb{R}^2\big)$$

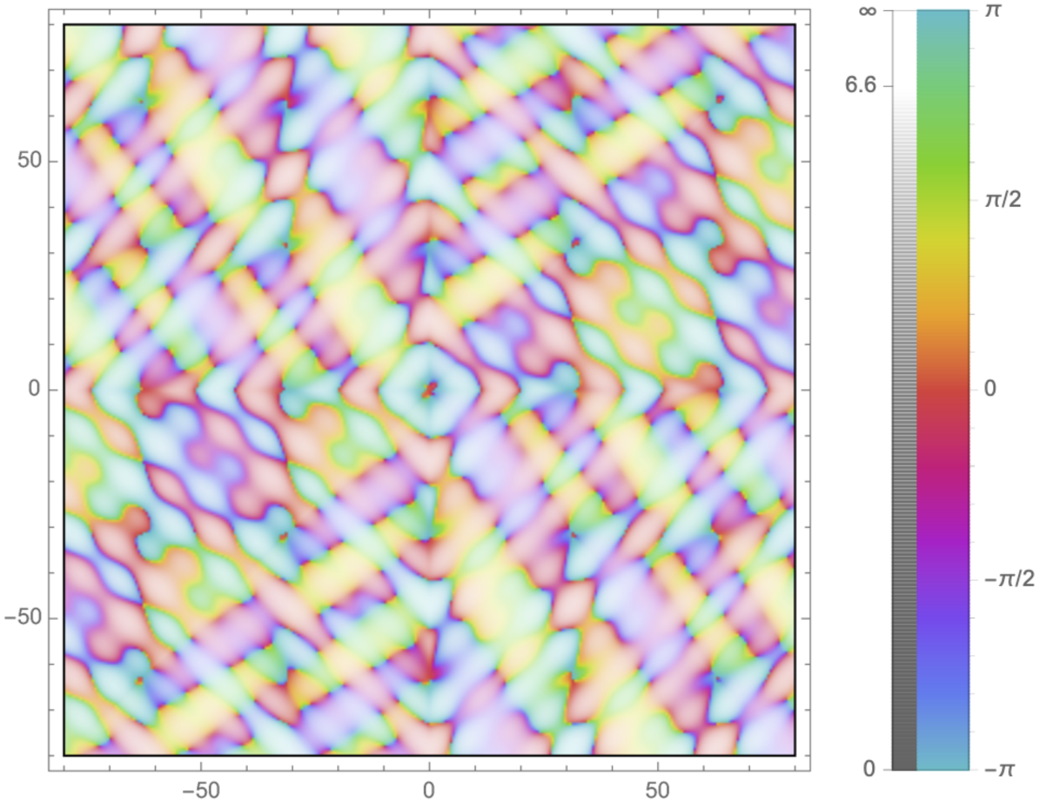


$$\mathcal{E}\mathcal{F}[\mathbf{1}_S] : \mathbb{R}^2 \to \mathbb{C}$$

Toy example : square minus a crack



$$\mathbf{1}_S - \mathbf{1}_C \in \mathrm{CF}\big(\mathbb{R}^2\big)$$



$$\mathcal{E}\mathcal{F}[\mathbf{1}_S - \mathbf{1}_C] : \mathbb{R}^2 \to \mathbb{C}$$

Persistent magnitude [14]

Graded pers. mod. over $\mathbb{R}$

$$M = \bigoplus_{j \in \mathbb{Z}} M_j$$

$$M_j \simeq \bigoplus_k \mathbf{k}_{[a_k^j, b_k^j)}$$

(finitely presented)

$$\longmapsto$$

Persistent magnitude function

$$|M| : \begin{aligned} \mathbb{R}_{>0} &\to \mathbb{R} \\ t &\mapsto \sum_{j,k} (-1)^j \left(e^{-t a_k^j} - e^{-t b_k^j} \right) \end{aligned}$$

---

[14] Govc, Hepworth (2021) *Persistent magnitude*. Journal of Pure and Applied Algebra, 225(3), 106517.

Persistent magnitude [14]

Graded pers. mod. over $\mathbb{R}$

$$M = \bigoplus_{j\in\mathbb{Z}} M_j$$

$$M_j \simeq \bigoplus_{k} \mathbf{k}_{[a^j_k, b^j_k)}$$

(finitely presented)

$$\longmapsto$$

Persistent magnitude function

$$|M| : \begin{aligned} \mathbb{R}_{>0} &\to \mathbb{R} \\ t &\mapsto \sum_{j,k} (-1)^j \left(e^{-ta^j_k} - e^{-tb^j_k} \right) \end{aligned}$$

Main property : additivity

If $0 \to M \to N \to P \to 0$ s.e.s., then $|N| = |M| + |P|$.

Csq. $|\mathrm{PH}(X, f)|(t) = \sum_{p\in\mathrm{Crit(f)}} (-1)^{\mu(p)} e^{-tf(p)}$

---

[14] Govc, Hepworth (2021) *Persistent magnitude*. Journal of Pure and Applied Algebra, 225(3), 106517.

Persistent magnitude [14]

Graded pers. mod. over $\mathbb{R}$

$$M = \bigoplus_{j\in\mathbb{Z}} M_j$$

$$M_j \simeq \bigoplus_{k} \mathbf{k}_{[a_k^j, b_k^j)}$$

(finitely presented)

Persistent magnitude function

$$|M| : \begin{aligned} \mathbb{R}_{>0} &\to \mathbb{R} \\ t &\mapsto \sum_{j,k} (-1)^j \left(e^{-ta_k^j} - e^{-tb_k^j}\right) \\ &= \int_{\mathbb{R}} e^{-s} \underbrace{t_*\varphi_M(s)}_{\varphi_M(s/t)} \mathrm{d}s \end{aligned}$$

$$\varphi_M : t \mapsto \textstyle\sum_{j\in\mathbb{Z}} (-1)^j \dim M_j(t) = \sum_{j,k} (-1)^j \mathbf{1}_{[a_k^j, b_k^j)}$$

Constructible function over $\mathbb{R}$

$$\begin{tikzcd}
M \arrow[rr, mapsto] \arrow[dr, mapsto] & & {|M|} \\
& \varphi_M \arrow[ur, mapsto] &
\end{tikzcd}$$

---

[14] Govc, Hepworth (2021) *Persistent magnitude*. Journal of Pure and Applied Algebra, 225(3), 106517.

Persistent magnitude [14]

$$\begin{tikzcd}[column sep=small]
\text{Graded pers. mod. over } \mathbb{R}^{n} & & \text{Persistent magnitude function} \\
\begin{array}{c} M = \bigoplus_{j\in\mathbb{Z}} M_j \\ \text{constructible, compactly supported} \end{array} \arrow[rr, mapsto] \arrow[dr, mapsto] & & |M| : \begin{array}{rcl} \mathbb{R}^{n}_{>0} & \to & \mathbb{R} \\ \xi & \mapsto & \int_{\mathbb{R}} e^{-s} \xi_{*} \varphi_M(s)\,\mathrm{d}s \end{array} \Bigg| = \mathcal{E}\mathcal{L}[\varphi_M] \\
& \begin{array}{c} \varphi_M : x \mapsto \sum_{j\in\mathbb{Z}} (-1)^{j} \dim M_j(x) \\ \text{Constructible function over } \mathbb{R}^{n} \end{array} \arrow[ur, mapsto] &
\end{tikzcd}$$

---

[14] Govc, Hepworth (2021) *Persistent magnitude*. Journal of Pure and Applied Algebra, 225(3), 106517.

Persistent magnitude [14]

Graded pers. mod. over $\mathbb{R}^n$

Persistent magnitude function

$$\begin{tikzcd}[column sep=small] M = \bigoplus_{j\in\mathbb{Z}} M_j \arrow[rr, mapsto] \arrow[dr, mapsto] & & |M| : \left. \begin{array}{l} \mathbb{R}^n_{>0} \to \mathbb{R} \\ \xi \mapsto \int_{\mathbb{R}} e^{-s}\xi_*\varphi_M(s)\,\mathrm{d}s \end{array} \right| = \mathcal{E}\mathcal{L}[\varphi_M] \\ & \varphi_M : x \mapsto \sum_{j\in\mathbb{Z}}(-1)^j \dim M_j(x) \arrow[ur, mapsto] & \end{tikzcd}$$

constructible, compactly supported

Constructible function over $\mathbb{R}^n$

Properties

- Additivity
- Formula with homological critical values for $\mathrm{PH}(X, f)$
- Compatibility with constructible operations (convolution, pushforward, ...)

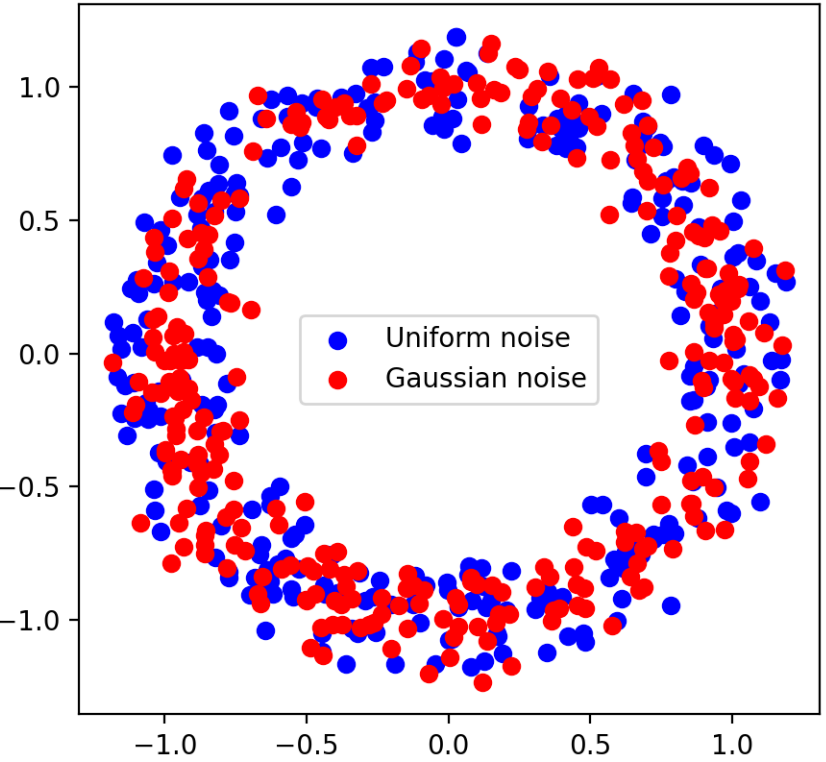
---

[14] Govc, Hepworth (2021) *Persistent magnitude*. Journal of Pure and Applied Algebra, 225(3), 106517.

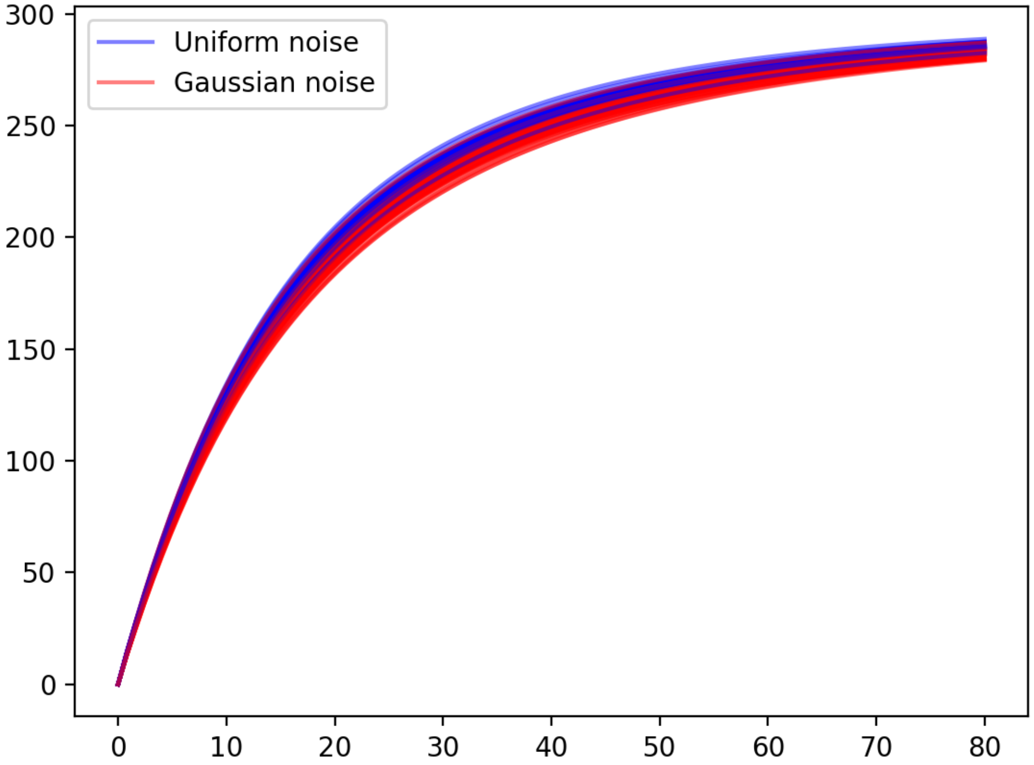
Toy example : persistence (with O. Hacquard)

$$X_1 = \mathcal{U}(\mathbb{S}^1) + 0.2 \cdot \mathcal{U}([-1,1]^2) \mapsto M_i = \bigoplus_{j \in \mathbb{Z}} H_j(\mathrm{Rips}(X_i)) \mapsto \varphi_i : t \mapsto \chi\left(\mathrm{Rips}_t(X_i)\right)$$

$$X_2 = \mathcal{U}(\mathbb{S}^1) + \mathcal{N}(0, 0.1 \cdot \mathrm{Id})$$



An example of point clouds



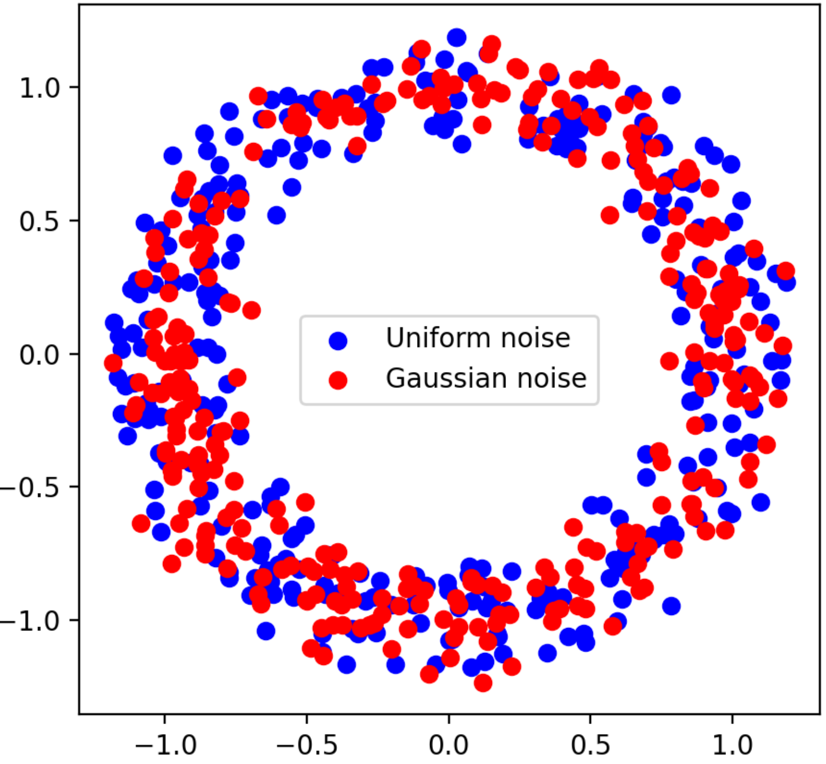
Persistent magnitude of 50 draws

$$\mathcal{EL}[\varphi_i] : t \mapsto \int_{\mathbb{R}} e^{-s} \varphi_i(s/t) \mathrm{d}s$$

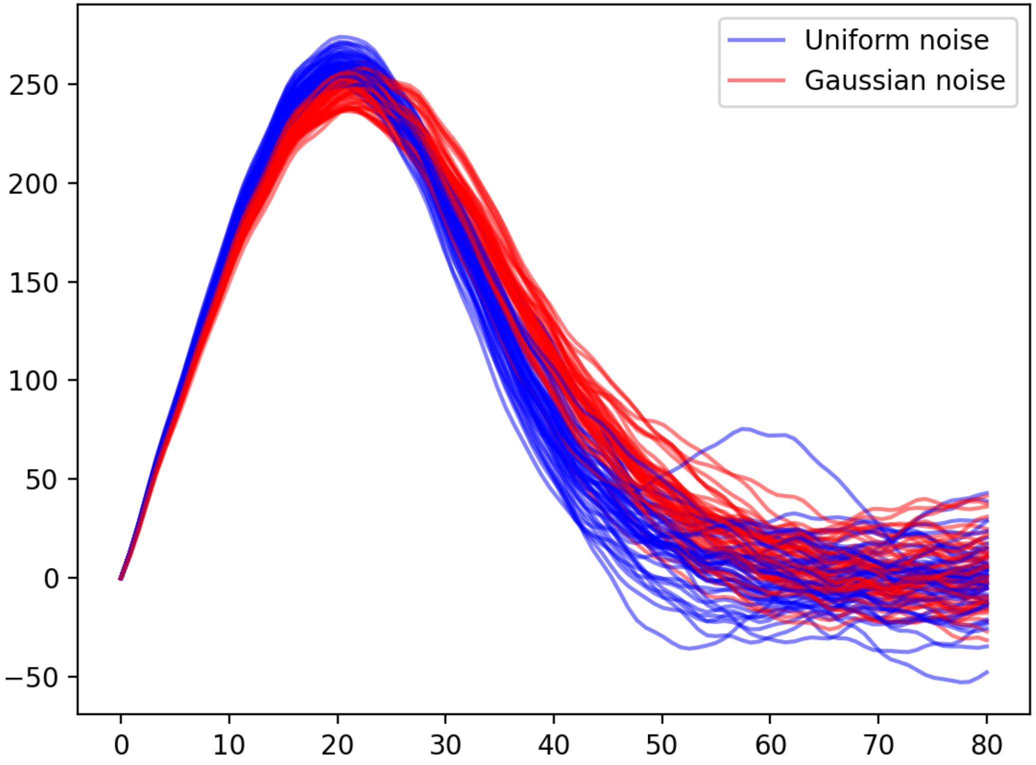
Toy example : persistence (with O. Hacquard)

$$X_1 = \mathcal{U}(\mathbb{S}^1) + 0.2 \cdot \mathcal{U}([-1,1]^2) \mapsto M_i = \bigoplus_{j\in\mathbb{Z}} H_j(\mathrm{Rips}(X_i)) \mapsto \varphi_i : t \mapsto \chi\left(\mathrm{Rips}_t(X_i)\right)$$

$$X_2 = \mathcal{U}(\mathbb{S}^1) + \mathcal{N}(0, 0.1 \cdot \mathrm{Id})$$



An example of point clouds



Hybrid Cosine transform of 50 draws

$$\mathrm{T}_{\cos}[\varphi_i] : t \mapsto \int_{\mathbb{R}} \cos(s)\varphi_i(s/t)\mathrm{d}s$$

Properties

1. **Regularity**

$$\mathcal{E}\mathcal{F} : \mathrm{CF}_{\mathrm{PL}}(\mathbb{R}^n) \longrightarrow C^0_{b,ps}(\mathbb{R}^n, \mathbb{C}) \; \Big]$$

continuous
piecewise smooth
bounded

2. **Invariance**

Translation $x_0 \in \mathbb{R}^n$, denote $\tau_{x_0 *}\varphi(x) := \varphi(x - x_0)$

$$\forall \xi \in \mathbb{R}^n, \quad \mathcal{E}\mathcal{F}[\tau_{x_0 *}\varphi](\xi) = e^{-i\langle \xi ; x_0 \rangle} \mathcal{E}\mathcal{F}[\varphi](\xi)$$

Linear transformation $A \in GL_n(\mathbb{R})$, denote $A_*\varphi(x) := \varphi(A^{-1}x)$

$$\forall \xi \in \mathbb{R}^n, \quad \mathcal{E}\mathcal{F}[A_*\varphi](\xi) = \mathcal{E}\mathcal{F}[\varphi]\left({}^t\!A\,\xi\right)$$

3. **Left inverse**

One can recover φ from $\mathcal{E}\mathcal{F}[\varphi]$. (under some assumptions $\supset$ persistence)

many others for other operations (e.g. constructible convolution)...

Summary

Hybrid transforms : Constructible world $\longrightarrow$ Usual functional spaces

e.g. $$\mathcal{E}\mathcal{F} : \mathrm{CF}_{\mathrm{PL}}(\mathbb{R}^n) \longrightarrow C^0_{b,ps}(\mathbb{R}^n, \mathbb{C})$$

topological information

Invariant of multi-parameter persistent modules $M \mapsto \varphi_M \mapsto \mathrm{T}_\kappa[\varphi_M]$

Properties

- Regularity
- Invariance (compatibility with operations)
- Computability

Summary

Hybrid transforms : Constructible world $\longrightarrow$ Usual functional spaces

e.g. $$\mathcal{E}\mathcal{F} : \mathrm{CF}_{\mathrm{PL}}(\mathbb{R}^n) \longrightarrow C^0_{b,ps}(\mathbb{R}^n, \mathbb{C})$$

topological information

Invariant of multi-parameter persistent modules $M \mapsto \varphi_M \mapsto \mathrm{T}_\kappa[\varphi_M]$

Properties

- Regularity
- Invariance (compatibility with operations)
- Computability

Future work

- Interpretability (with O. Hacquard)
- Stability

Summary

Hybrid transforms : Constructible world $\longrightarrow$ Usual functional spaces

e.g. $$\mathcal{E}\mathcal{F} : \mathrm{CF}_{\mathrm{PL}}(\mathbb{R}^n) \longrightarrow C^0_{b,ps}(\mathbb{R}^n, \mathbb{C})$$

topological information

Invariant of multi-parameter persistent modules $M \mapsto \varphi_M \mapsto \mathrm{T}_\kappa[\varphi_M]$

Properties

- Regularity
- Invariance (compatibility with operations)
- Computability

Future work

- Interpretability (with O. Hacquard)
- Stability

Thank you !

References

[1] Crawford, Monod, Chen, Mukherjee, Rabadán (2020) *Predicting Clinical Outcomes in Glioblastoma : An Application of Topological and Functional Data Analysis*, Journal of the American Statistical Association, 115 :531, 1139-1150

[2] Baryshnikov, Ghrist (2008) *Target enumeration via integration over planar sensor networks*. Proceedings of Robotics : Science and Systems IV, 127.

[3] Ghrist, Robinson (2011) *Euler–Bessel and Euler–Fourier transforms*. Inverse problems, 27(12), 124006.

[4] Schapira (1995) *Tomography of constructible functions.* In International Symposium on Applied Algebra, Algebraic Algorithms, and Error-Correcting Codes (pp. 427-435). Springer, Berlin, Heidelberg.

[5] Kashiwara, Schapira (1990) *Sheaves on Manifolds* (Vol. 292). Springer Science & Business Media.

[6] Curry, Ghrist, Robinson (2012) *Euler calculus with applications to signals and sensing.* Proceedings of Symposia in Applied Mathematics. Vol. 70.

[7] Curry, Mukherjee, Turner. (2018) *How many directions determine a shape and other sufficiency results for two topological transforms.* arXiv preprint arXiv :1805.09782.

[8] Govc, Hepworth (2021) *Persistent magnitude*. Journal of Pure and Applied Algebra, 225(3), 106517.

[9] Bobrowski, Borman (2012) *Euler integration of Gaussian random fields and persistent homology*. Journal of Topology and Analysis, 4(01), 49-70.

[10] Beltramo, Andreeva, Giarratano, Bernabeu, Sarkar, Skraba (2021) *Euler Characteristic Surfaces* arXiv preprint :2102.08260

[11] Turner, Mukherjee, Boyer (2014) *Persistent homology transform for modeling shapes and surfaces*. Information and Inference : A Journal of the IMA, 3(4), 310-344.

[12] Ghrist, Levanger, Mai (2018) *Persistent homology and Euler integral transforms*. Journal of Applied and Computational Topology, 2(1), 55-60.

[13] Meng, Anand, Lu, Wu, Xia (2020) *Weighted persistent homology for biomolecular data analysis*. Scientific reports, 10(1), 1-15.

[14] Govc, Hepworth (2021) *Persistent magnitude*. Journal of Pure and Applied Algebra, 225(3), 106517.