Euler-Fourier transform of constructible functions

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Abstract

We present an integral transform of constructible functions, the *Euler-Fourier transform*, combining Lebesgue integration and constructible pushforward — a topological dimensionality reduction. Lebesgue integration gives access to regularity results while constructible pushforward conveys topological information, making it strictly more discriminating than the classical Fourier transform. This transform is an example of the more general notion of hybrid transform defined in [8]. In this note, we adapt the exposition to this specific example and illustrate it in various ways. We also show that it can be efficiently computed in practical scenarios.

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1 Introduction

Euler calculus – the integral calculus of constructible functions with respect to the Euler characteristic – is of increasing interest in topological data analysis and computational geometry. Already in [9], it was developed as an alternative definition of convolution for polygonal tracings with multiplicities, a useful notion in robotics [5, 7]. In persistence theory, Schapira's result on Radon transform [11] positively answers an important question [2, Thm. 4.11]: are two constructible subsets of $\mathbb{R}^n$ with the same persistent homology in all degrees and for all height filtrations equal? More generally, the constructible functions naturally associated to multiparameter persistent modules stand as simple and well-behaved, albeit incomplete, invariants of these objects. For instance, the *persistent magnitude* [4] is actually defined on the constructible functions associated to the persistence modules.

In [8], we introduced a general definition and conducted a systematic study of integral transforms combining Lebesgue integration and Euler calculus for constructible functions. Such transforms generalize the *Bessel* and *Fourier* transforms of Ghrist and Robinson [3], as well as the *Euler characteristic of barcodes* of Bobrowski and Borman [1]. In this note, we illustrate the theory on one example, the *Euler-Fourier transform*. We state some of its characteristics and illustrate its differences from its classical analogue in various situations (see Figure 1). More general results are proven in [8].

2 Definition

A function $\varphi : \mathbb{R}^n \to \mathbb{Z}$ is called *constructible*[1] if it can be written as a finite sum $\varphi = \sum_{i=1}^{r} m_i \mathbf{1}_{K_i}$, where the m_i's are integers and the K_i's are compact subanalytic subsets of $\mathbb{R}^n$. We denote by $\mathrm{CF}(\mathbb{R}^n)$ the group of constructible functions on $\mathbb{R}^n$. We refer to [6, Sec. 8.2, Sec. 9.7] for more details on subanalytic sets and constructible functions.

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[1] In this note, we consider only compactly supported constructible functions.

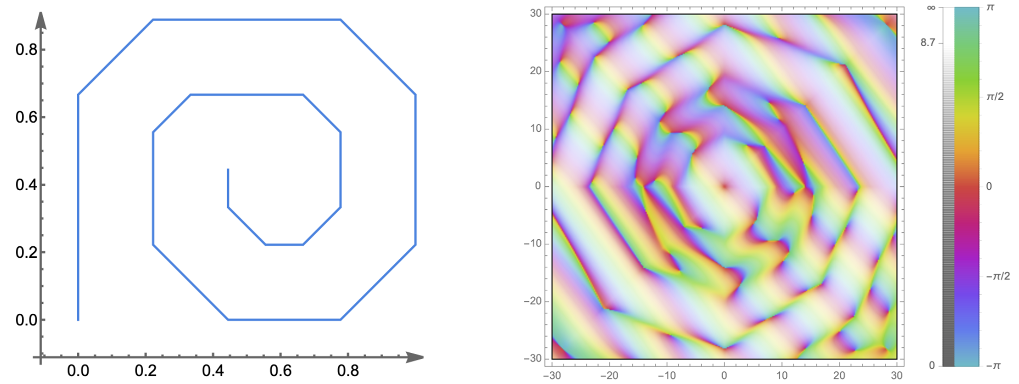


Figure 1 Left: a piecewise-linear closed curve C in $\mathbb{R}^2$. Right: Euler-Fourier transform of $\mathbf{1}_C$.

▶ **Example 2.1.** Any polytope of $\mathbb{R}^n$ — the convex hull of a finite set of points of $\mathbb{R}^n$ — is subanalytic. If the subsets K_i in the decomposition of φ are polytopes, then φ is said *PL-constructible*. We denote by $\mathrm{CF}_{\mathrm{PL}}(\mathbb{R}^n)$ the group of PL-constructible functions on $\mathbb{R}^n$.

▶ **Definition 2.2.** Let $\xi \in \mathbb{R}^n$ and $\varphi = \sum_{i=1}^r m_i \mathbf{1}_{K_i}$ be a constructible function. The *pushforward of* φ *along* ξ is the constructible function $\xi_*\varphi$ over $\mathbb{R}$ defined for any $t \in \mathbb{R}$ by

$$\xi_*\varphi(t) = \sum_{i=1}^{r} m_i \cdot \chi\left(\xi^{-1}(t) \cap K_i\right),$$

where[2] $\xi^{-1}(t) = \{x \in \mathbb{R}^n \,;\, \langle \xi; x\rangle = t\}$ and χ is the Euler characteristic, that is $\chi(Z) = \sum_{j\in\mathbb{Z}}(-1)^j \dim_{\mathbb{Q}} H^j(Z;\mathbb{Q})$ for any $Z \subseteq \mathbb{R}^n$ compact and subanalytic. See Figure 2.

The fact that this definition does not depend on the decomposition of φ and that $\xi_*\varphi$ is a constructible function on $\mathbb{R}$ is proven by Schapira [9, 10].

▶ **Example 2.3.** If $P \subseteq \mathbb{R}^n$ is a polytope, then $\xi_*\mathbf{1}_P = \mathbf{1}_{[\min_P(\xi),\max_P(\xi)]}$, where $\min_P(\xi) = \min\{\langle \xi; x\rangle \,;\, x \in P\}$ and $\max_P(\xi) = \max\{\langle \xi; x\rangle \,;\, x \in P\}$. In fact, there is a vertex p (resp. q) of P, depending on ξ, such that $\min_P(\xi) = \langle \xi; p\rangle$ (resp. $\max_P(\xi) = \langle \xi; q\rangle$).

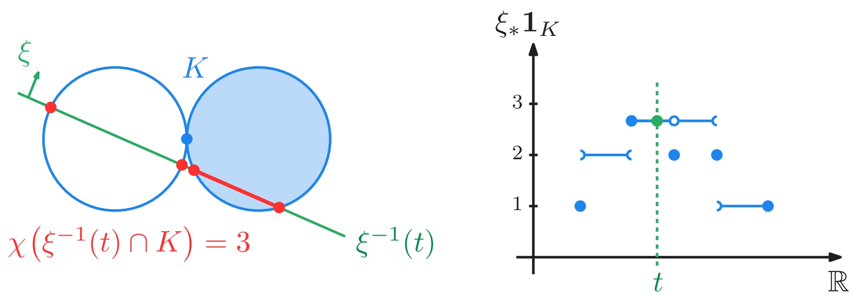


Figure 2 The pushforward along $\xi \in \mathbb{R}^2$ of $\mathbf{1}_K$ for the compact subanalytic $K \subseteq \mathbb{R}^2$.

[2] For any two $x, y \in \mathbb{R}^n$, we denote by $\langle x; y\rangle$ their canonical scalar product.

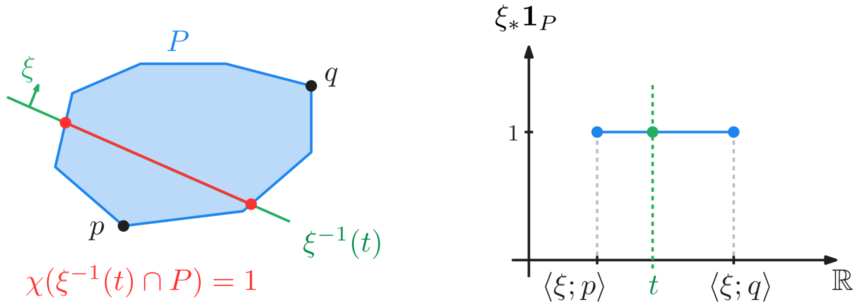


Figure 3 The pushforward along $\xi \in \mathbb{R}^2$ of $\varphi = \mathbf{1}_P$ for the polytope $P \subseteq \mathbb{R}^2$.

▶ **Definition 2.4.** The *Euler-Fourier transform* of $\varphi \in \mathrm{CF}(\mathbb{R}^n)$ is defined for $\xi \in \mathbb{R}^n$ by:

$$\mathcal{EF}[\varphi](\xi) = \int_{\mathbb{R}} e^{-it} \xi_* \varphi(t)\, \mathrm{d}t.$$

Choosing any kernel $\kappa \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ instead of $t \mapsto e^{-it}$ leads to the general definition of hybrid transform, studied in [8]. We now turn to examples. The reader's attention is drawn to the effect of the successive application of topological pushforward and of classical integral.

▶ **Example 2.5.** Denote by $\mathbb{S}_r$ the sphere of radius $r > 0$ in $\mathbb{R}^n$. For any $\xi \in \mathbb{R}^n$,

$$\mathcal{EF}[\mathbf{1}_{\mathbb{S}_r}](\xi) = 2 \cdot (1 + (-1)^n) \cdot \sin(r\|\xi\|).$$

▶ **Example 2.6.** Consider the constructible function $\varphi = \mathbf{1}_S - \mathbf{1}_C$, where $S = [-1/2, 1/2]^2$ and C is the piecewise linear closed curve of $\mathbb{R}^2$ represented by the dotted line in Figure 4b. Since C has zero volume, the (classical) Fourier transforms of $\mathbf{1}_S$ and of $\mathbf{1}_S - \mathbf{1}_C$ are equal. However, their Euler-Fourier transforms differ, as shown in Figure 4.

3 Properties

The Euler-Fourier transform enjoys a regularity result on PL-constructible functions.

▶ **Proposition 3.1.** *Let $\varphi \in \mathrm{CF}_{\mathrm{PL}}(\mathbb{R}^n)$. The function $\mathcal{EF}[\varphi]$ is continuous, bounded and piecewise smooth on $\mathbb{R}^n$.*

The Euler-Fourier transform enjoys several invariance properties. We emphasize here specific ones which are also satisfied by the classical Fourier transform.

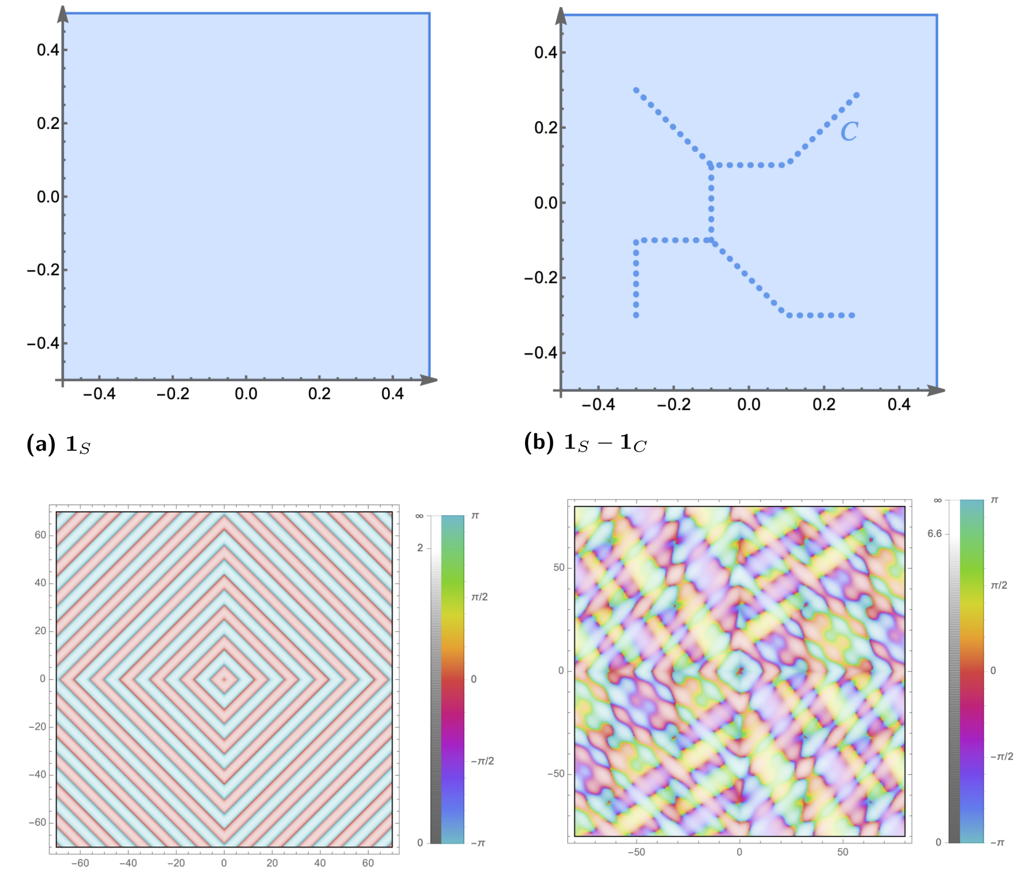
▶ **Proposition 3.2.** *Let $\varphi \in \mathrm{CF}(\mathbb{R}^n)$ and $A \in \mathrm{GL}_n(\mathbb{R})$. Denote by $A_*\varphi$ the constructible function on $\mathbb{R}^n$ given by $A_*\varphi(x) = \varphi(A^{-1}x)$, for any $x \in \mathbb{R}^n$. For any $\xi \in \mathbb{R}^n$, we have:*

$$\mathcal{EF}[A_*\varphi](\xi) = \mathcal{EF}[\varphi]\left({}^t\!A\,\xi\right).$$

▶ **Proposition 3.3.** *Let $\varphi \in \mathrm{CF}(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$. Denote by $\tau_{x_0 *}\varphi$ the constructible function on $\mathbb{R}^n$ given by $\tau_{x_0 *}\varphi(x) = \varphi(x - x_0)$, for any $x \in \mathbb{R}^n$. For any $\xi \in \mathbb{R}^n$, we have:*

$$\mathcal{EF}[\tau_{x_0 *}\varphi](\xi) = e^{-i\langle \xi; x_0\rangle} \cdot \mathcal{EF}[\varphi](\xi).$$

These operations are not the only operations available on constructible functions. In [8], we study the compatibility of hybrid transforms with numerous operations.



(a) $\mathbf{1}_S$ (b) $\mathbf{1}_S - \mathbf{1}_C$

(c) $\mathcal{EF}\left[\mathbf{1}_S\right]$ (d) $\mathcal{EF}\left[\mathbf{1}_S - \mathbf{1}_C\right]$

Figure 4 Euler-Fourier transforms of the constructible functions $\mathbf{1}_S$ and $\mathbf{1}_S - \mathbf{1}_C$ in Example 2.6.

4 Computations

Let $\varphi \in \mathrm{CF}_{\mathrm{PL}}(\mathbb{R}^n)$ be written as $\varphi = \sum_{l=1}^r m_l \cdot \mathbf{1}_{P_l}$ where the subsets P_l are polytopes. By $\mathbb{Z}$-linearity of $\mathcal{EF}$ and Example 2.3, we have for any $\xi \in \mathbb{R}^n$,

$$\mathcal{EF}\left[\varphi\right](\xi) = \sum_{l=1}^{r} m_l \int_{\min_{P_l}(\xi)}^{\max_{P_l}(\xi)} e^{-it}\,\mathrm{d}t = i \sum_{l=1}^{r} m_l \left(e^{-i \max_{P_l}(\xi)} - e^{-i \min_{P_l}(\xi)}\right). \tag{4.1}$$

The extrema $\min_{P_l}(\xi)$ and $\max_{P_l}(\xi)$ being attained on vertices of P_l, computing the extrema of $\langle \xi; v_l \rangle$ for v_l ranging over the set of vertices of P_l yields the value of $\mathcal{EF}\left[\varphi\right](\xi)$ using (4.1).

Consider now a fixed finite collection of polytopes $\mathcal{P} = \{P_l\}_{l=1}^r$ and denote by $\mathrm{CF}_{\mathcal{P}}(\mathbb{R}^n)$ the set of $\varphi \in \mathrm{CF}(\mathbb{R}^n)$ that can be written as $\varphi = \sum_{l=1}^r m_l \cdot \mathbf{1}_{P_l}$. Precomputing the extrema of ξ on the set of vertices of each polytope of $\mathcal{P}$, the Euler-Fourier transform of any $\varphi \in \mathrm{CF}_{\mathcal{P}}(\mathbb{R}^n)$ is easily computed using (4.1). As an important example, greyscale images of size $n \times m$ can naturally be seen as constructible functions on a fixed cubical complex of $\mathbb{R}^2$. Each value of their transforms can thus be computed in $\mathcal{O}(nm)$ operations.

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