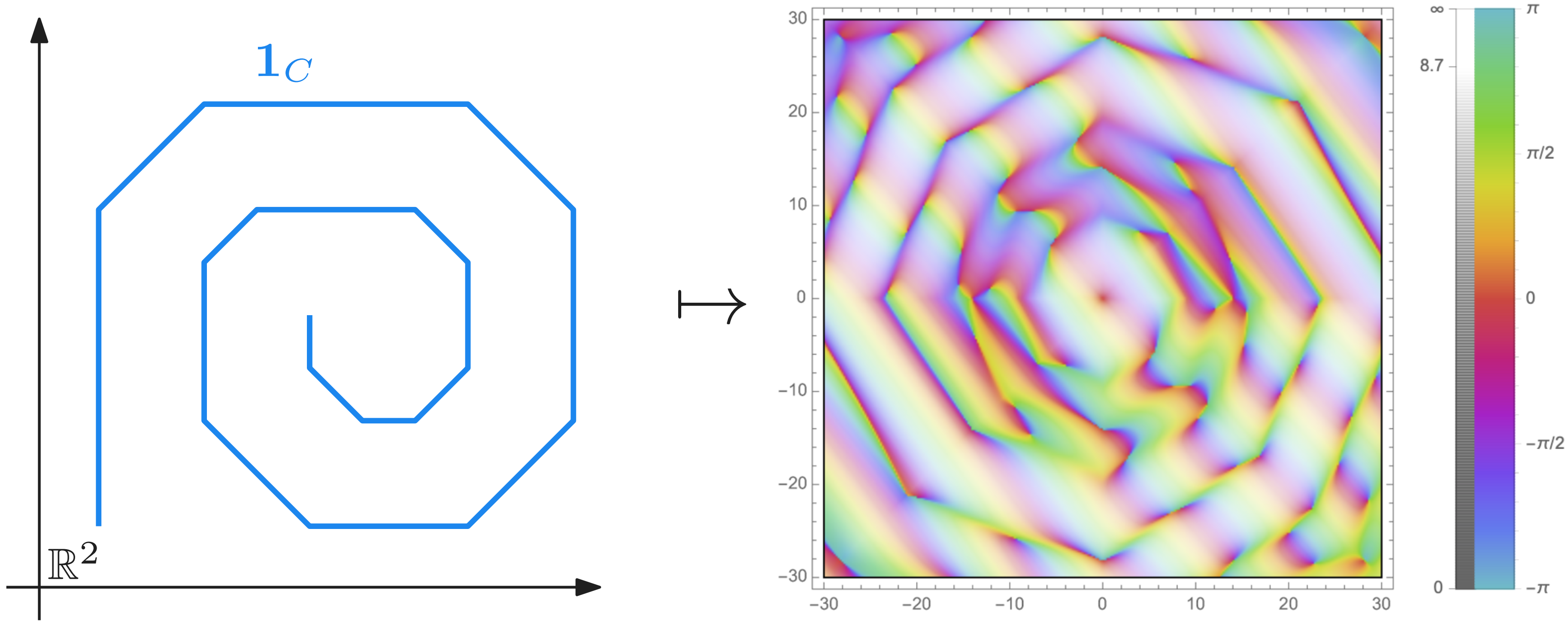
Euler-Fourier transform

Vadim Lebovici

arXiv:2111.07829

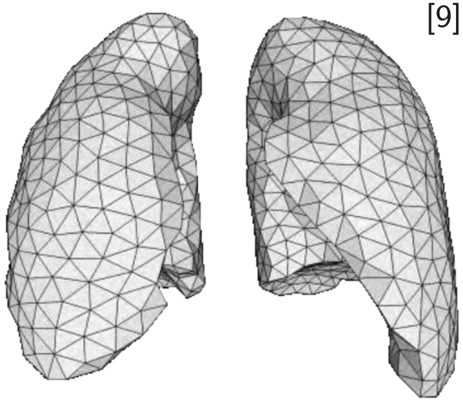


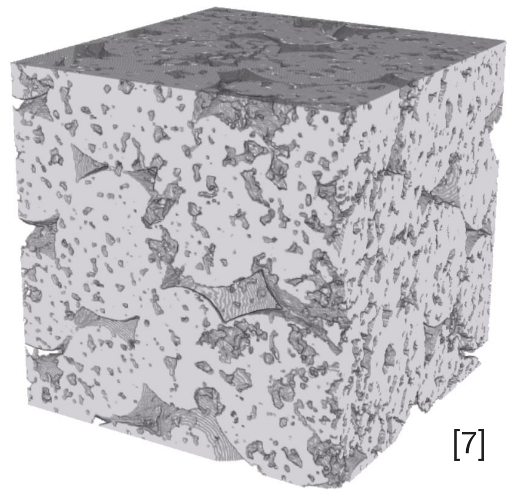
PhD supervised by Steve Oudot (INRIA) & François Petit (Université de Paris)

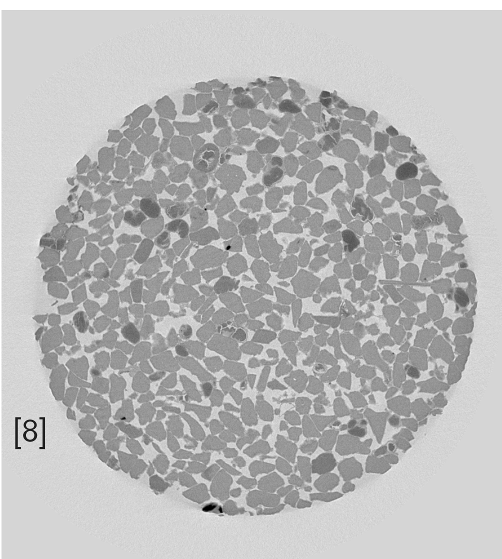
YRF Session 3B n°5 - 22/02/15

Introduction

Shapes → **Descriptors**


[9]


[7]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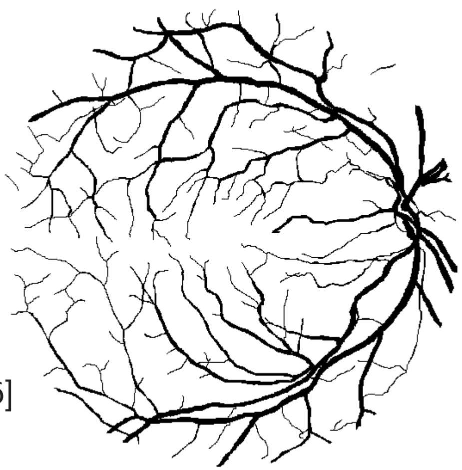

[6]

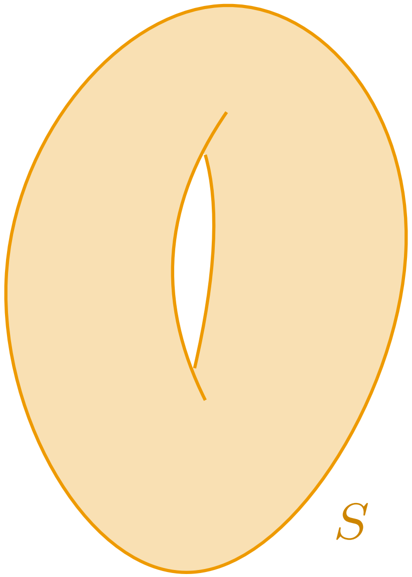
- topological meaning
- interpretable
- well-suited to statistics
- complete

↓

Analysis

Introduction

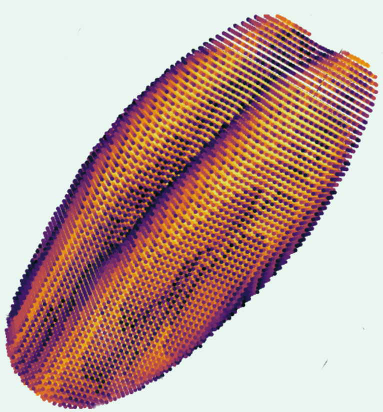
Shapes —— Topological integral transforms ——> **Descriptors**

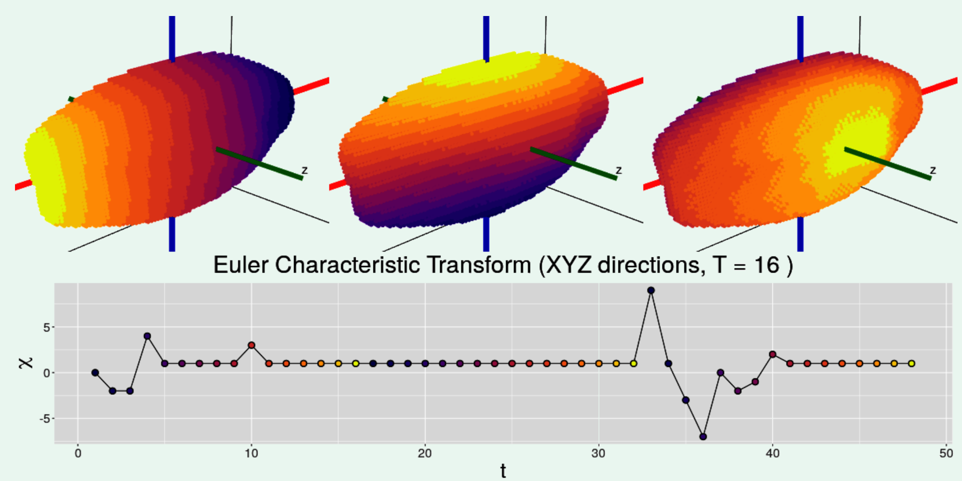


e.g. Radon transform [1], Euler characteristic transform [2,3,4]

- topological meaning
- interpretable
- well-suited to statistics
- complete

Ex. [12] Classification of barley seeds

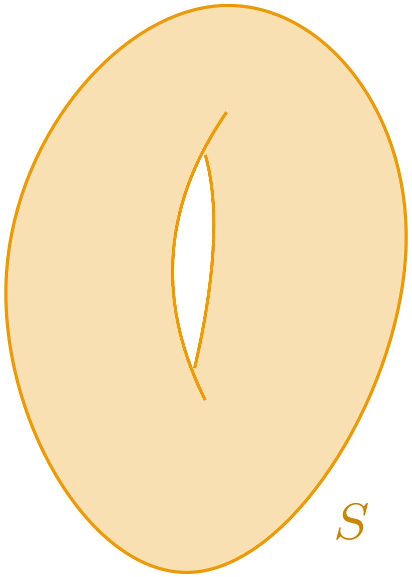




[12] Amézquita, Quigley, Ophelders, Landis, Koenig, Munch, Chitwood (2022) *Measuring hidden phenotype : Quantifying the shape of barley seeds using the Euler Characteristic Transform*. in silico Plants, 4(1).

Introduction

Shapes



Topological integral transforms

Hybrid transforms

mixed
classical + topological

Descriptors

- topological meaning
- interpretable
- well-suited to statistics
- complete

\+ diversity of kernels

\+ regularity

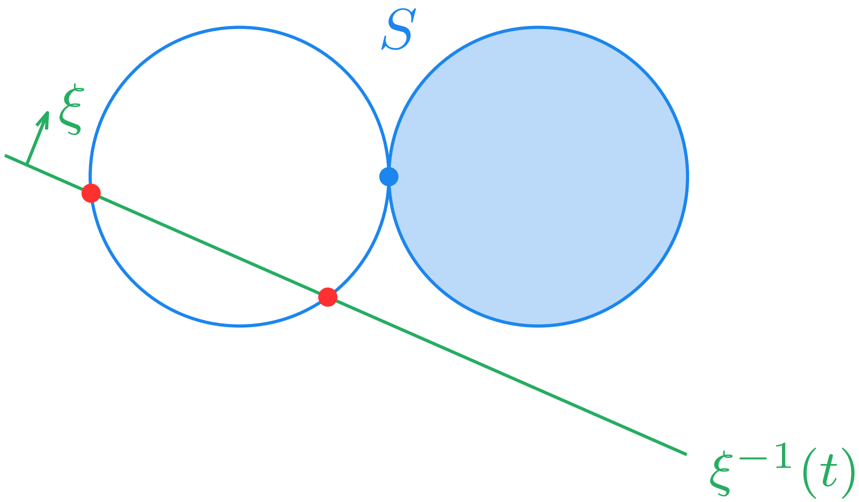
e.g. **Euler-Fourier transform**

Fourier analysis
of
topological changes

Radon transform (Schapira [1])

Consider $\xi : \mathbb{R}^n \to \mathbb{R}$ linear.

Ex.



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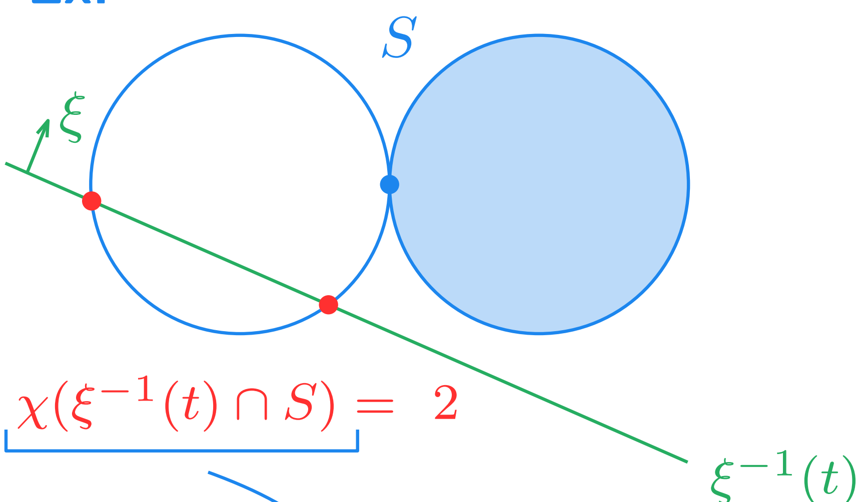

Def. Euler characteristic

$$\chi(S) = \sum_{j \in \mathbb{Z}} (-1)^j \dim H_j(S; \mathbb{Q})$$

compact subanalytic

$$\chi(S) = \sum_{j \in \mathbb{Z}} (-1)^j \#\{j\text{-simplices}\}$$ if S simp. cplx

$\chi(S) = 1$ if S compact convex

Csq. easy to compute !

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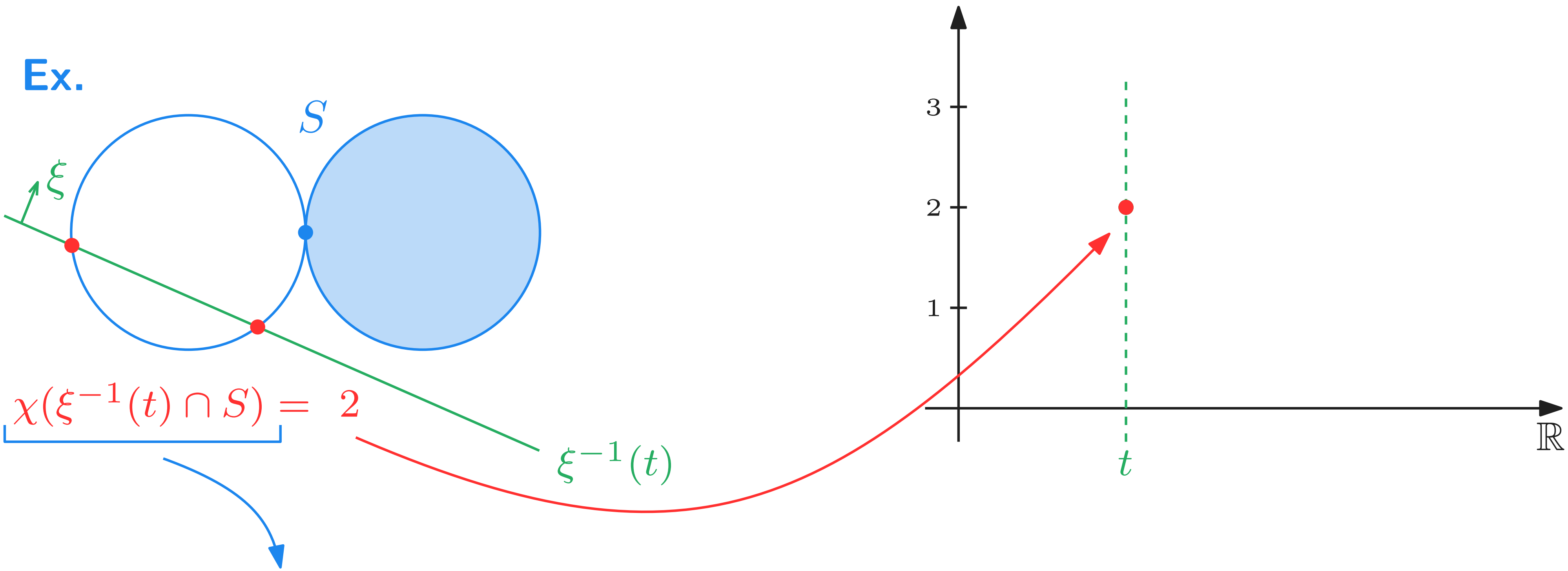

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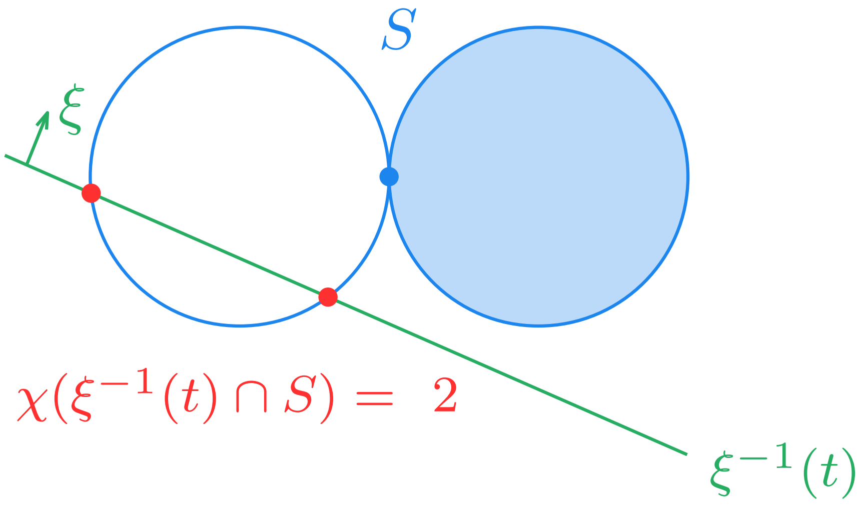
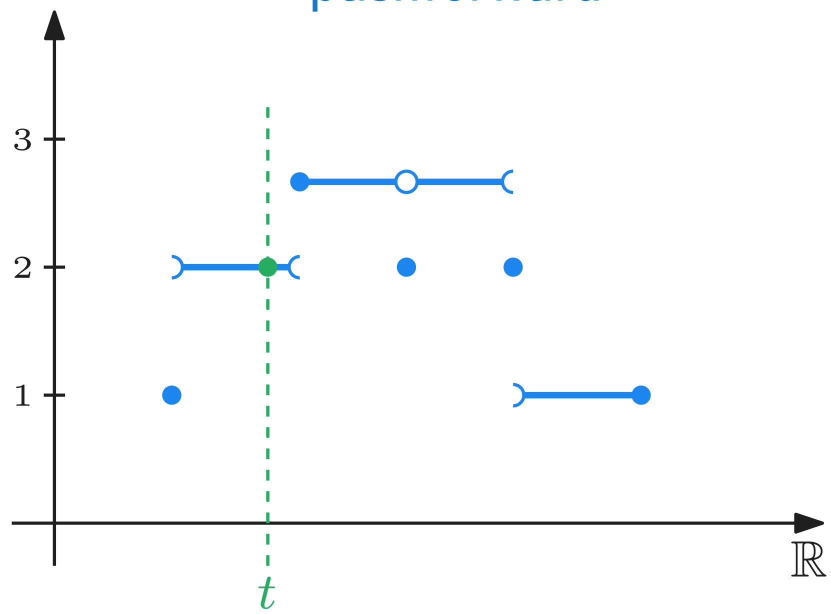


"pushforward"



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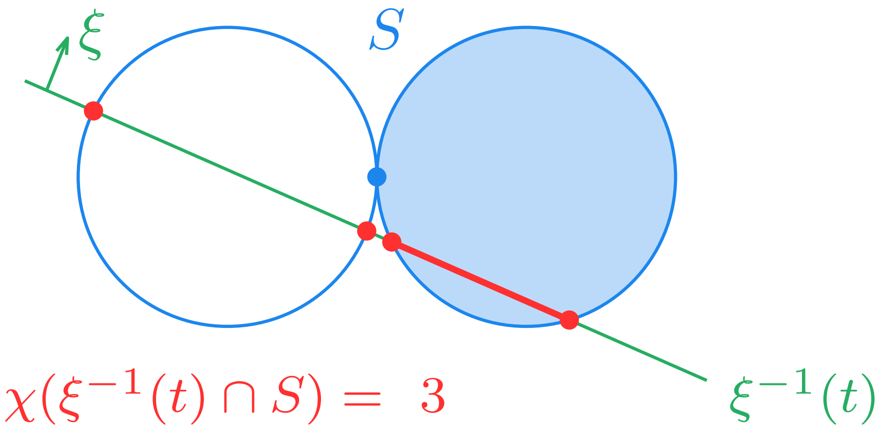

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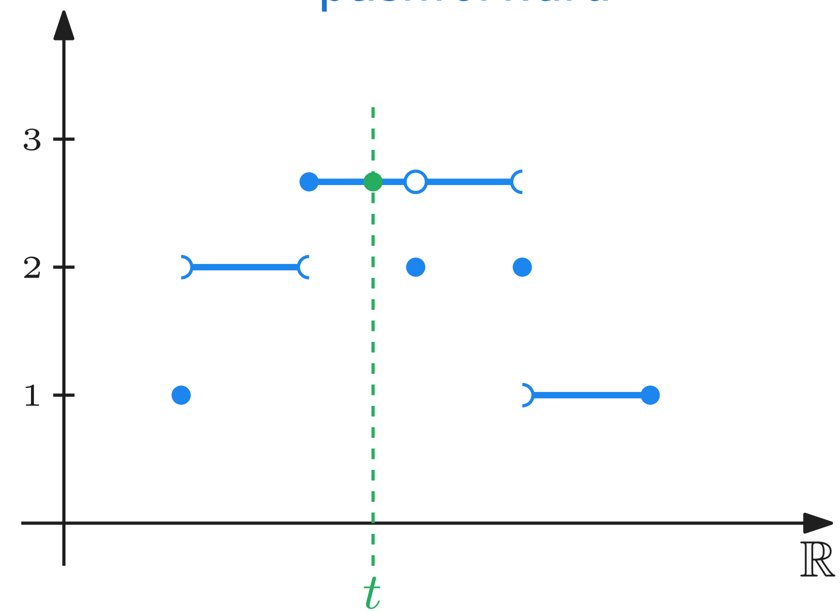

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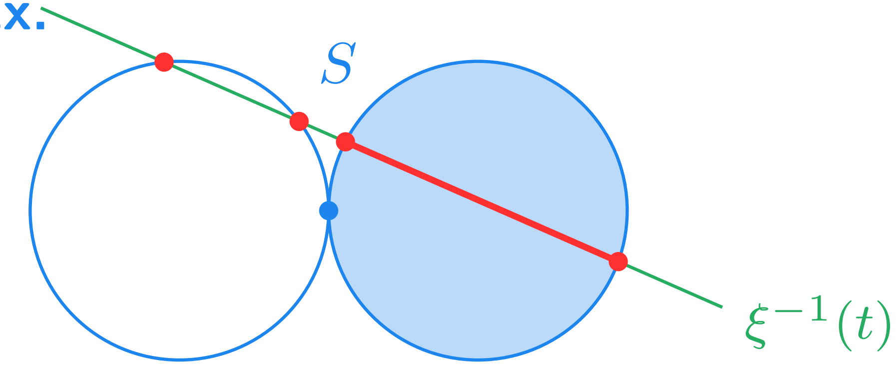
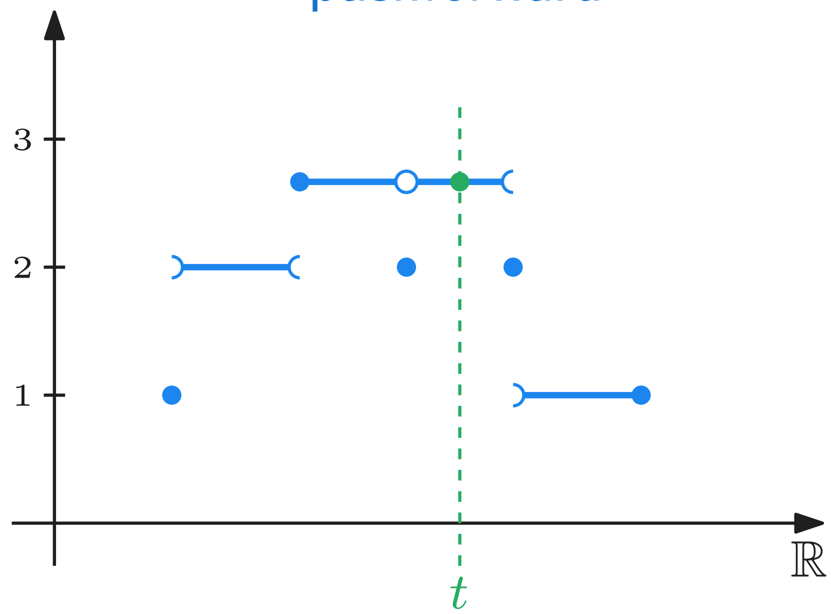


$\chi(\xi^{-1}(t) \cap S) = 3$

"pushforward"



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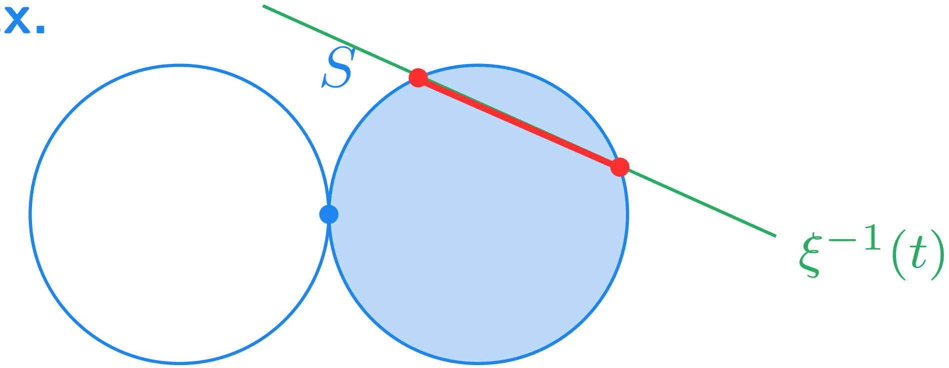
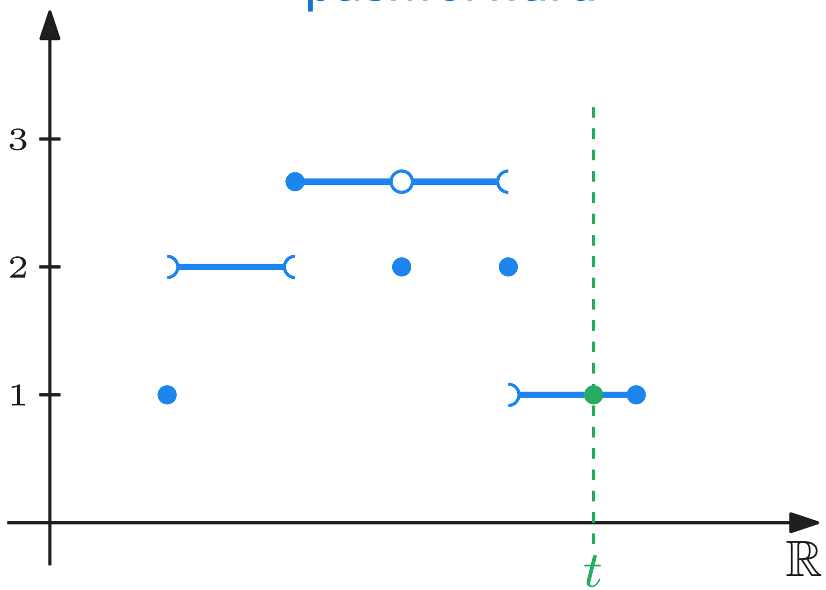


$\chi(\xi^{-1}(t) \cap S) = 1$

"pushforward"



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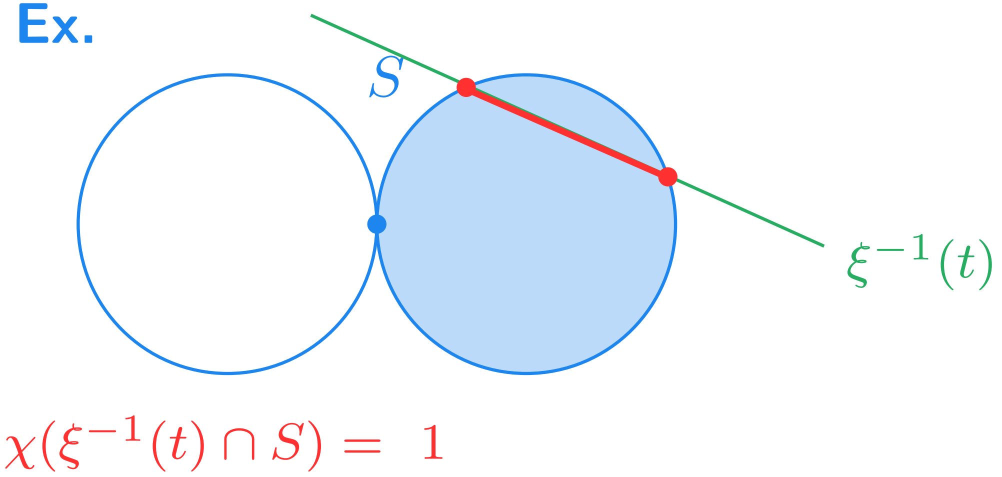

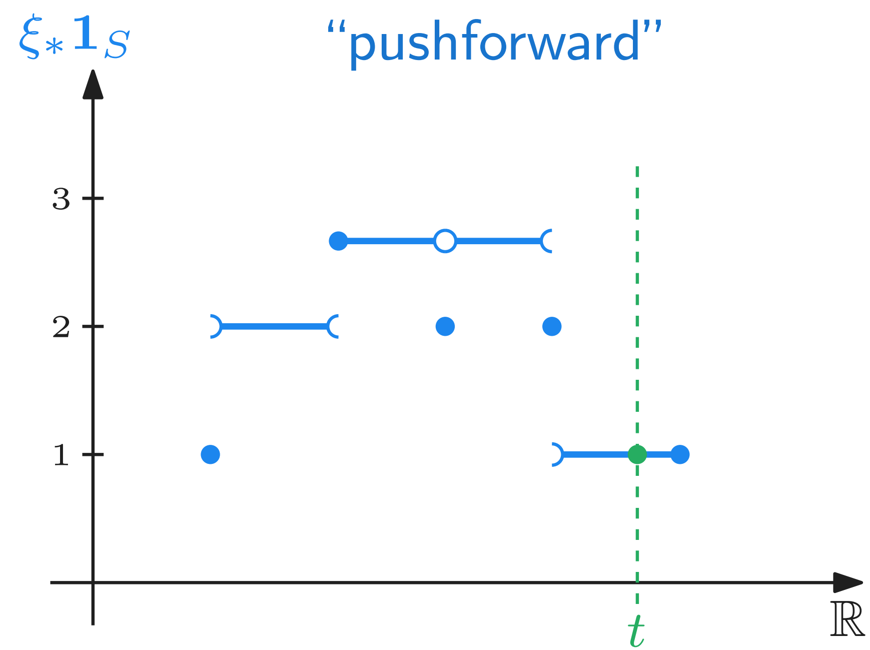

Def. (Pushforward) S compact subanalytic

$$\xi_* \mathbf{1}_S : \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{Z} \\ t \longmapsto \chi\left(\xi^{-1}(t) \cap S\right) \end{array}$$

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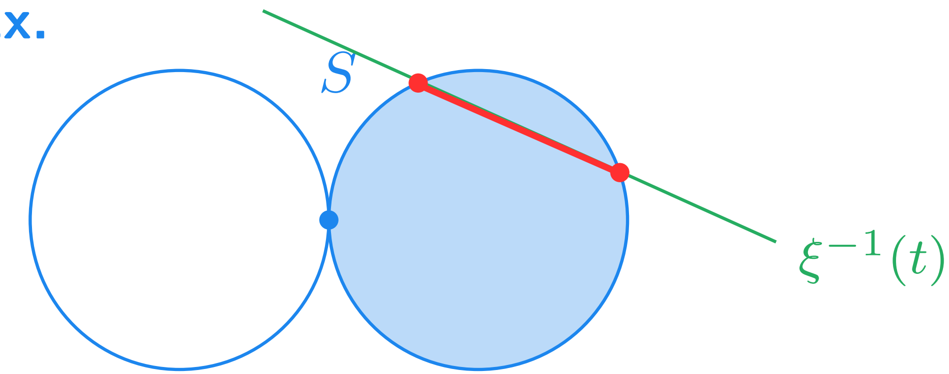

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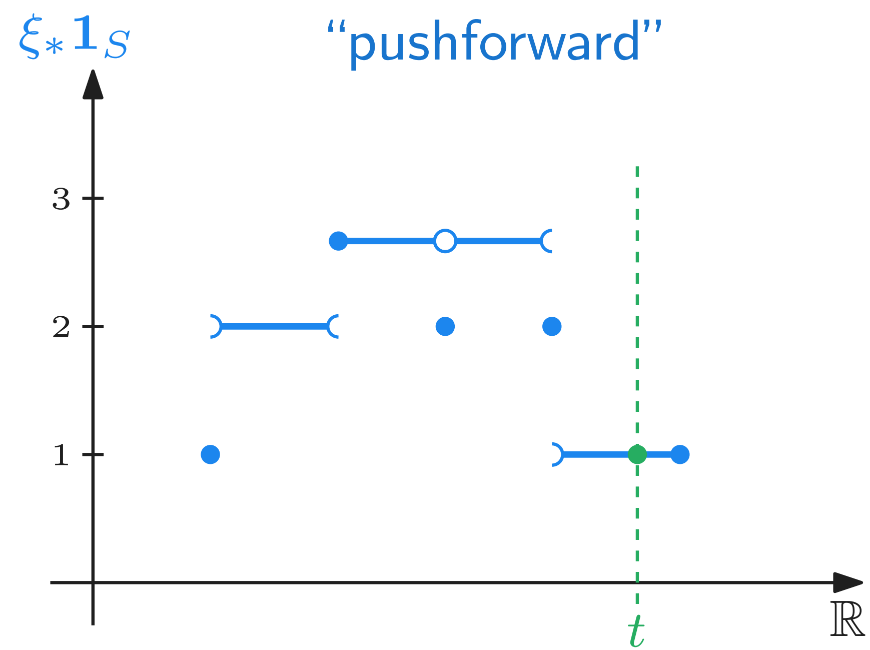

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Def. Radon transform

$$\mathcal{R}[S] : \begin{array}{rcl} \mathbb{S}^{n-1} \times \mathbb{R} & \longrightarrow & \mathbb{Z} \\ (\xi, t) & \longmapsto & \xi_* \mathbf{1}_S(t) \end{array}$$

Thm. (Schapira [1])
$S \mapsto \mathcal{R}[S]$ is injective (up to a constant if n is even).

Euler characteristic transform

[2] (Curry, Mukherjee, Turner 2018)
[3] (Turner, Mukherjee, Boyer 2014)
[4] (Ghrist, Levanger, Mai 2018)

Consider $\xi : \mathbb{R}^n \to \mathbb{R}$ linear.

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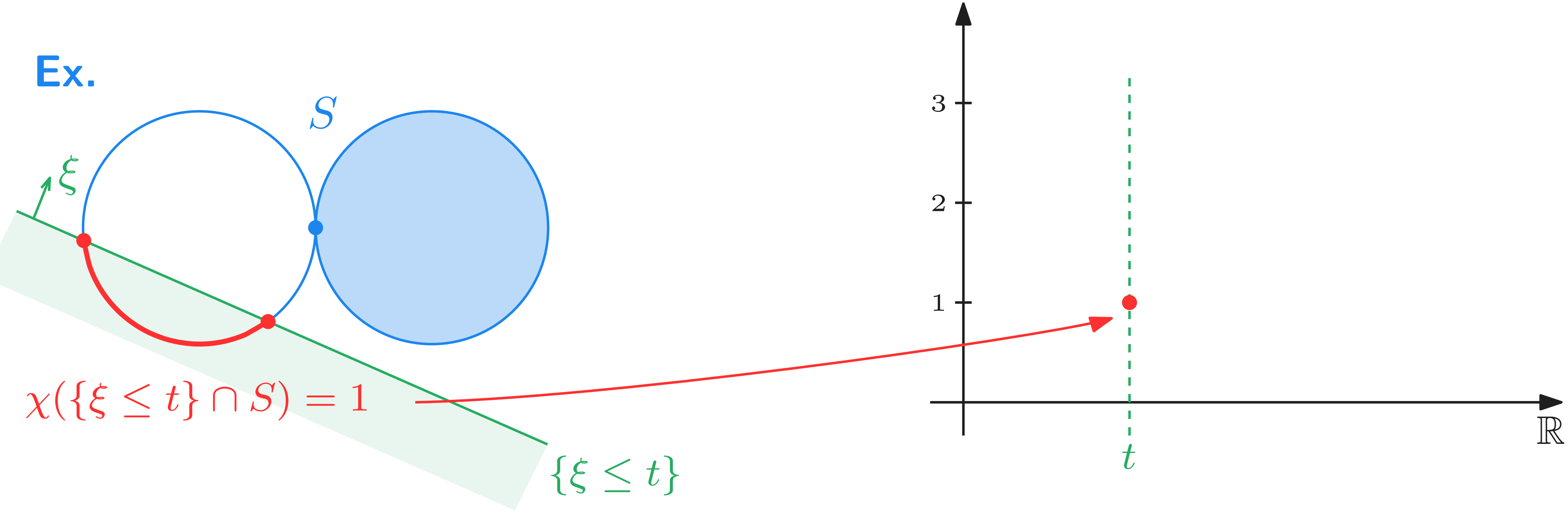

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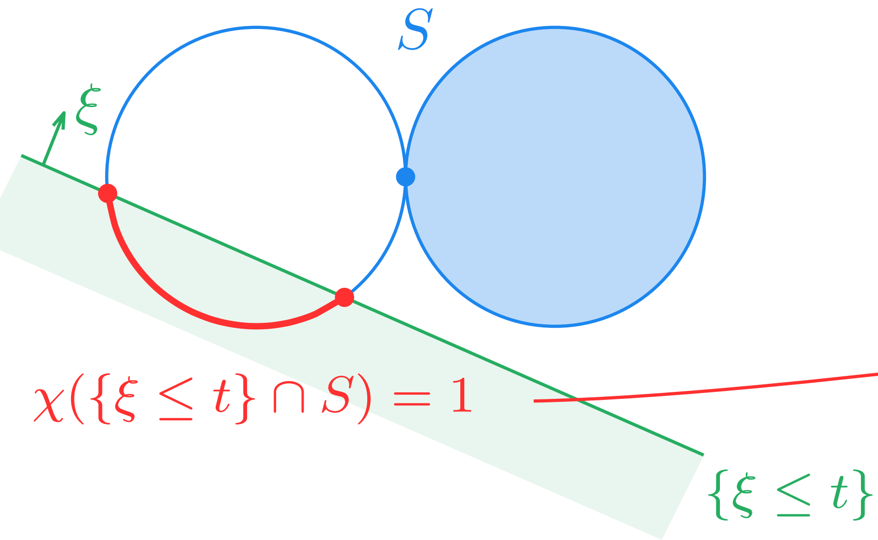

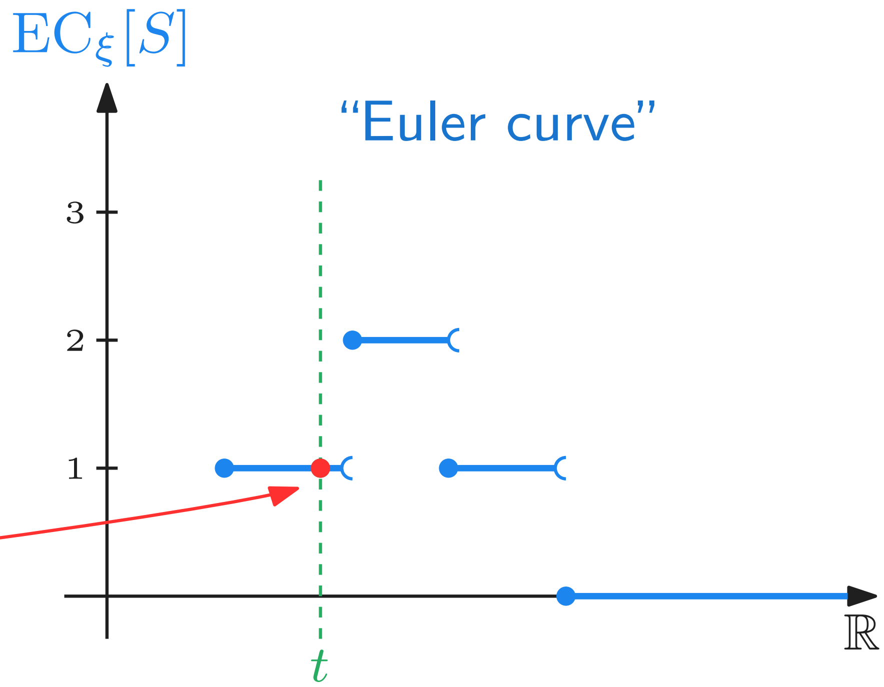

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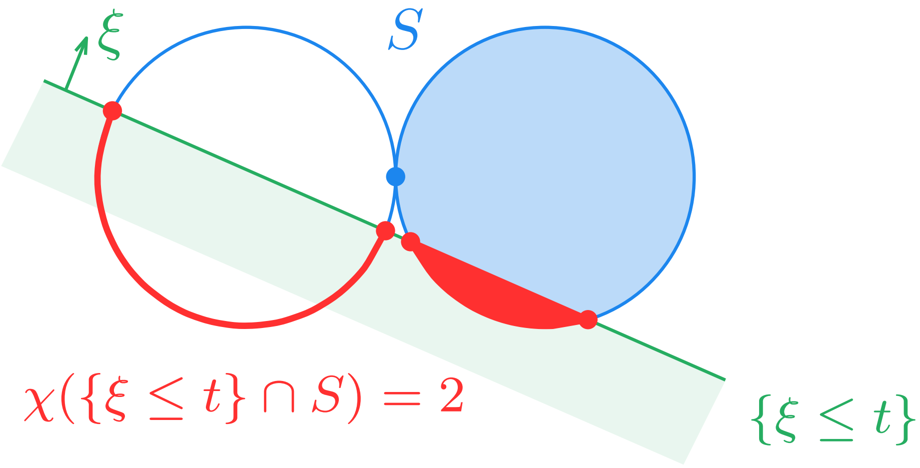

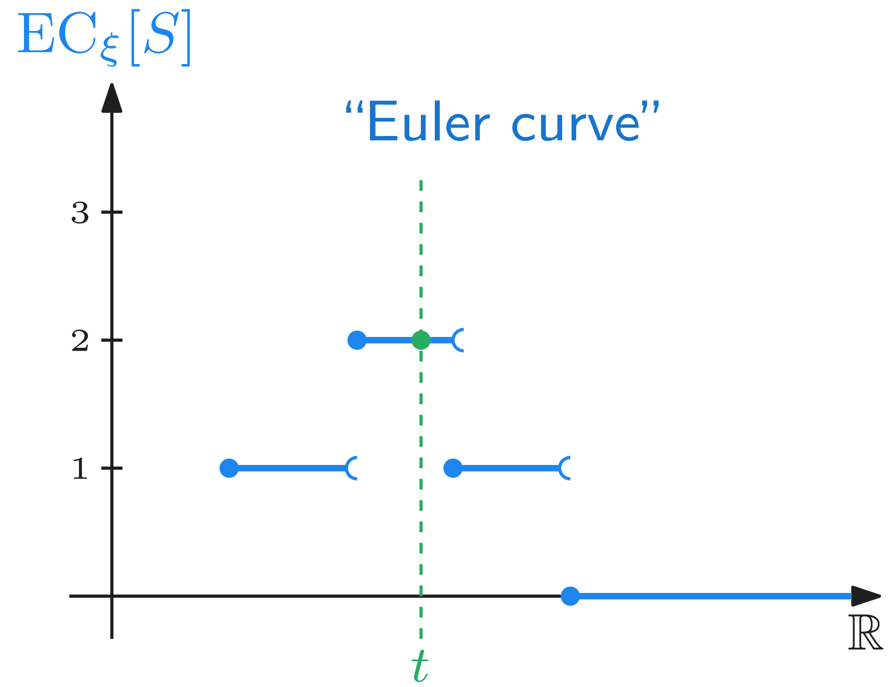

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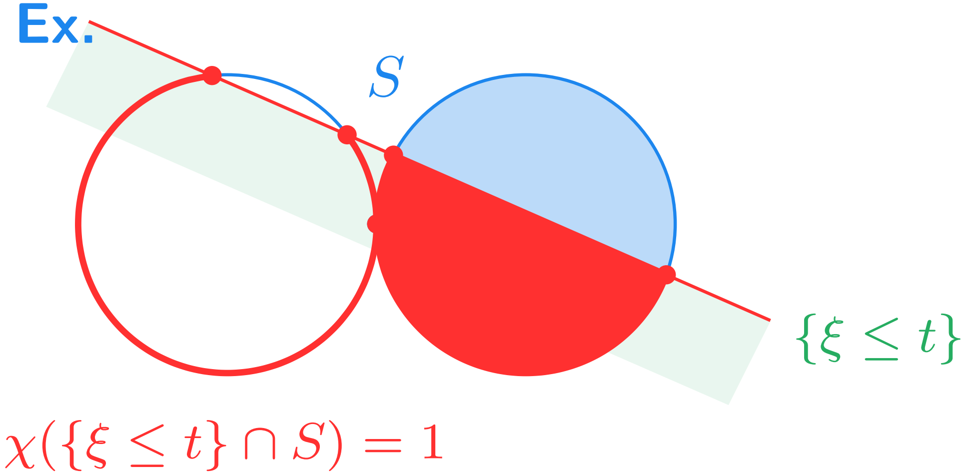

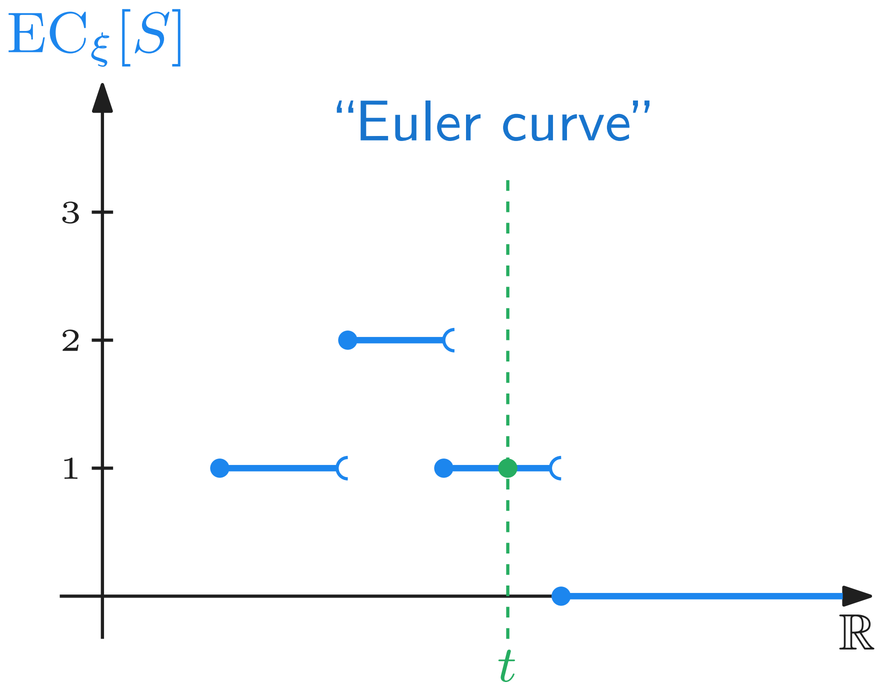

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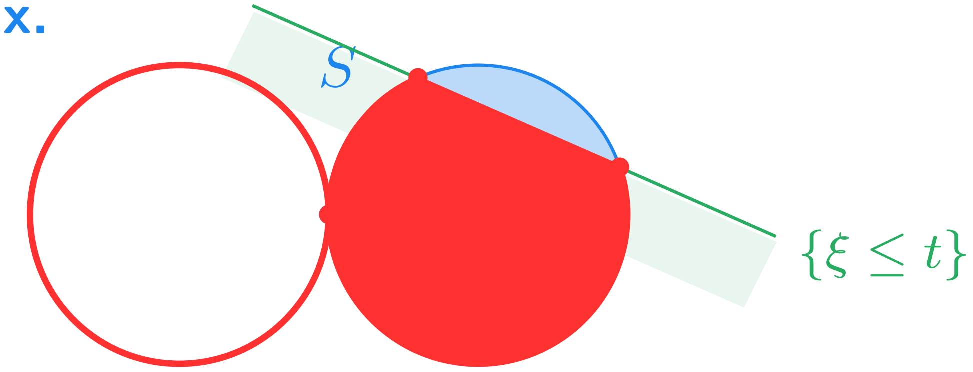

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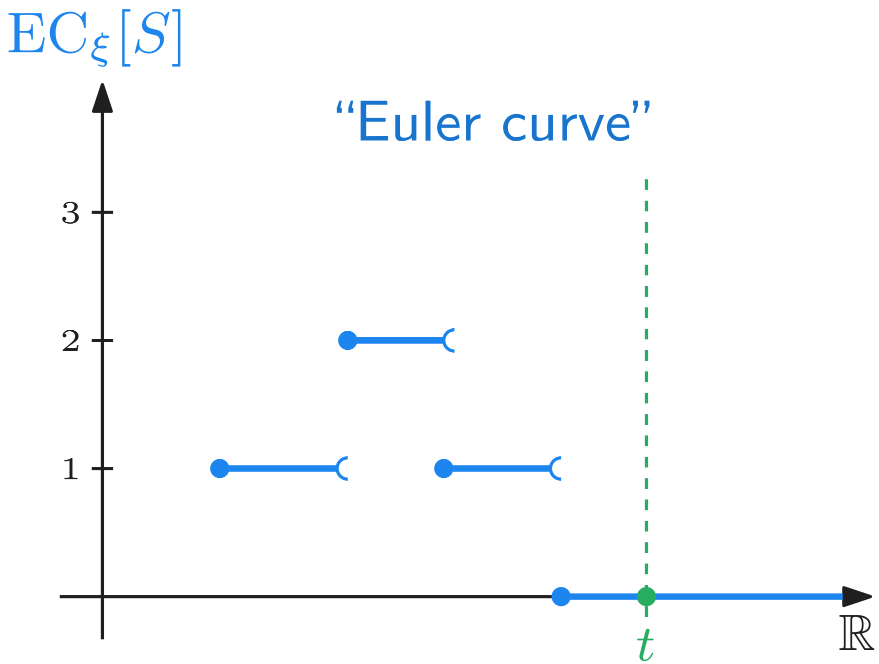

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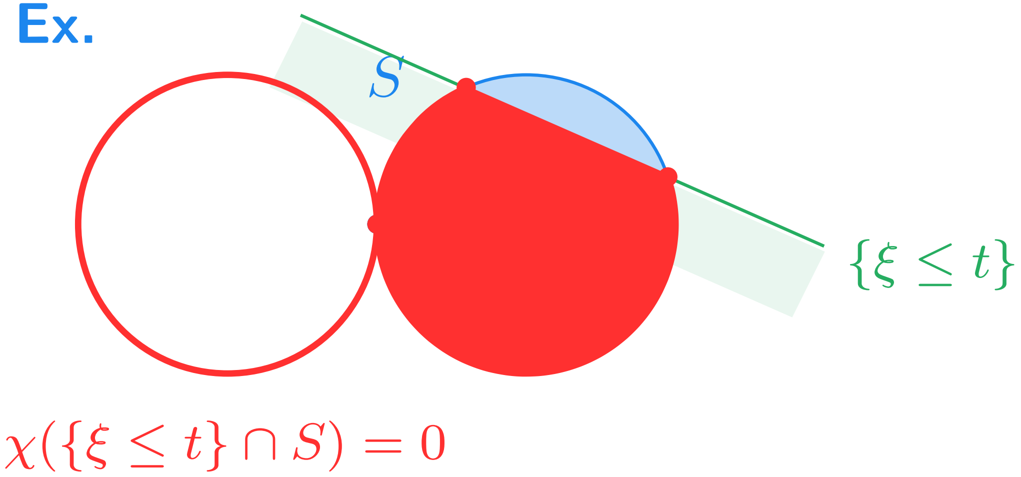

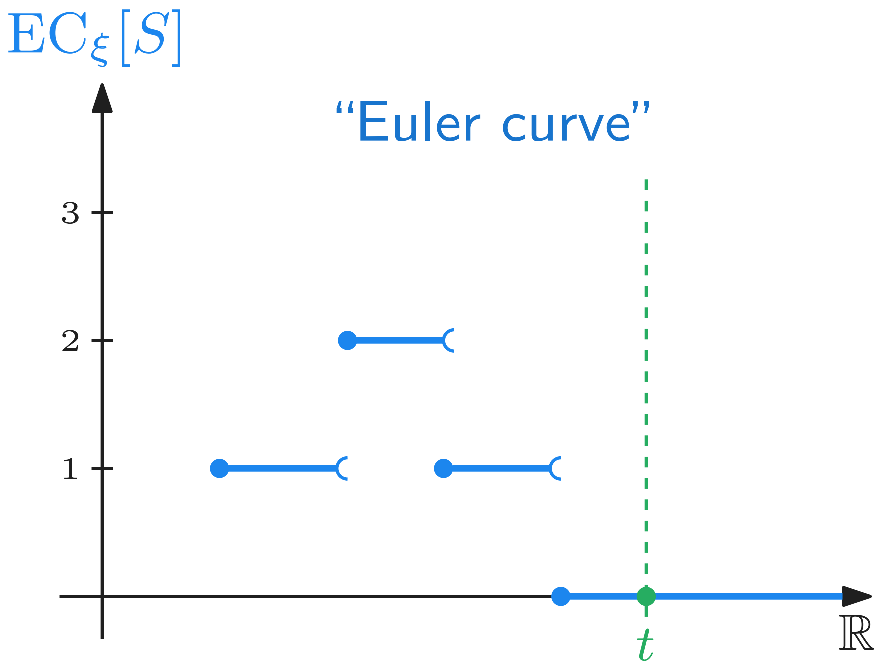

Def. Euler characteristic transform (ECT)

$$\mathrm{ECT}[S] : \begin{array}{ccc} \mathbb{S}^{n-1} \times \mathbb{R} & \longrightarrow & \mathbb{Z} \\ (\xi, t) & \longmapsto & \mathrm{EC}_\xi[S](t) \end{array}$$

Thm. (Curry, Mukherjee, Turner [2], Ghrist, Levanger, Mai [4])

$S \mapsto \mathrm{ECT}[S]$ is injective.

Euler characteristic transform

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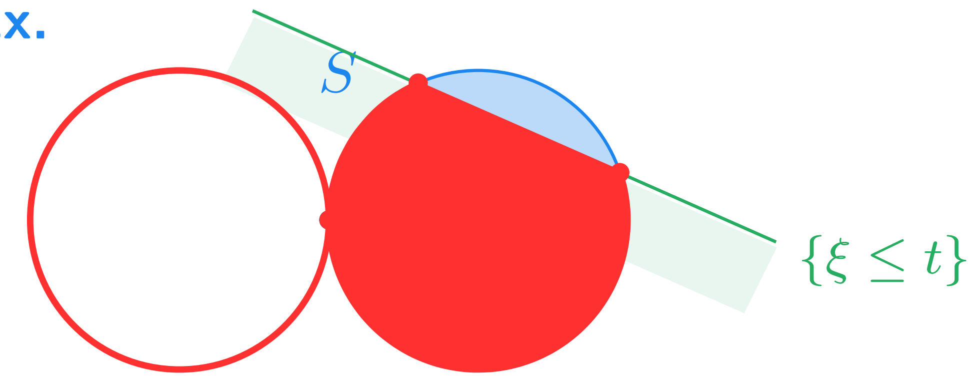

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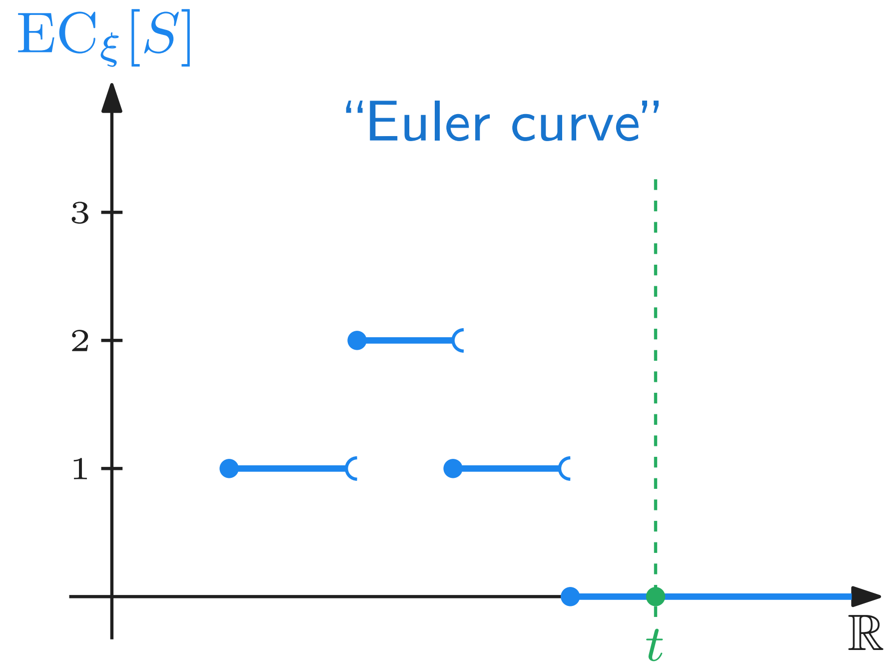
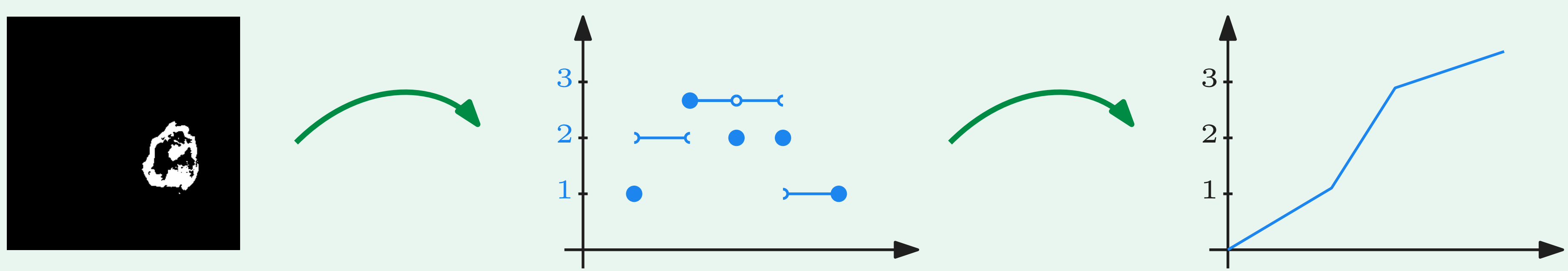


Ex. [10] Prediction of clinical outcomes in brain tumors



MRI | Euler curves | smoothed Euler curves

[10] Crawford, Monod, Chen, Mukherjee, Rabadán (2020) *Predicting Clinical Outcomes in Glioblastoma : An Application of Topological and Functional Data Analysis*, Journal of the American Statistical Association, 115 :531, 1139-1150

Hybrid transforms

Def. (Hybrid transform) $\kappa : \mathbb{R} \to \mathbb{C}$ in L^1_{loc} and S compact subanalytic

$$\mathrm{T}_\kappa[S] : \begin{array}{rcl} \mathbb{R}^n & \longrightarrow & \mathbb{C} \\ \xi & \longmapsto & \displaystyle\int_{\mathbb{R}} \kappa(t)\xi_* \mathbf{1}_S(t)\,\mathrm{d}t = \int_{\mathbb{R}} \kappa(t)\mathcal{R}[S](\xi, t)\,\mathrm{d}t \end{array}$$

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Def. (Hybrid transform) $\kappa : \mathbb{R} \to \mathbb{C}$ in L^1_{loc} and S compact subanalytic

integration against kernel

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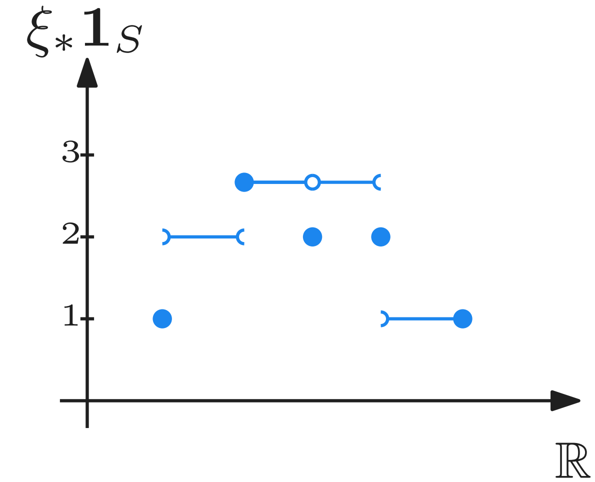
topological dim. reduction



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Ex. Euler-Fourier

$$\mathcal{E}\mathcal{F}[S] : \begin{array}{rcl} \mathbb{R}^n & \longrightarrow & \mathbb{C} \\ \xi & \longmapsto & \displaystyle\int_{\mathbb{R}} e^{-it}\xi_* \mathbf{1}_S(t)\,\mathrm{d}t \end{array}$$

Fourier analysis of topological changes

Ex. Euler-Laplace

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Multi-parameter persistent magnitude

generalizes persistent magnitude [11]

[11] Govc, Hepworth (2021) *Persistent magnitude*. Journal of Pure and Applied Algebra, 225(3), 106517.

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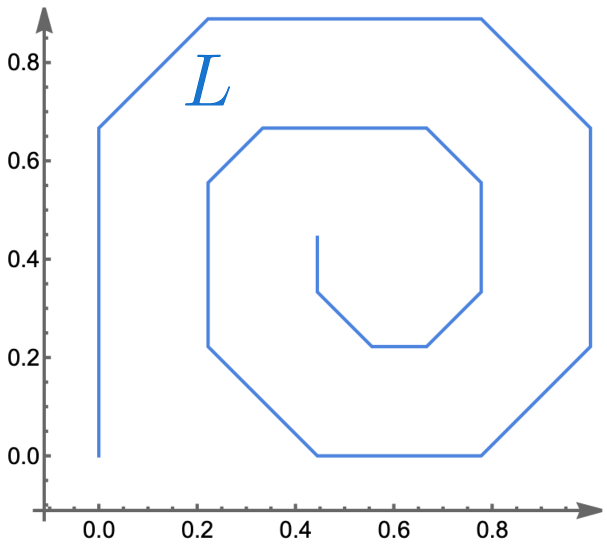
Euler-Fourier transform

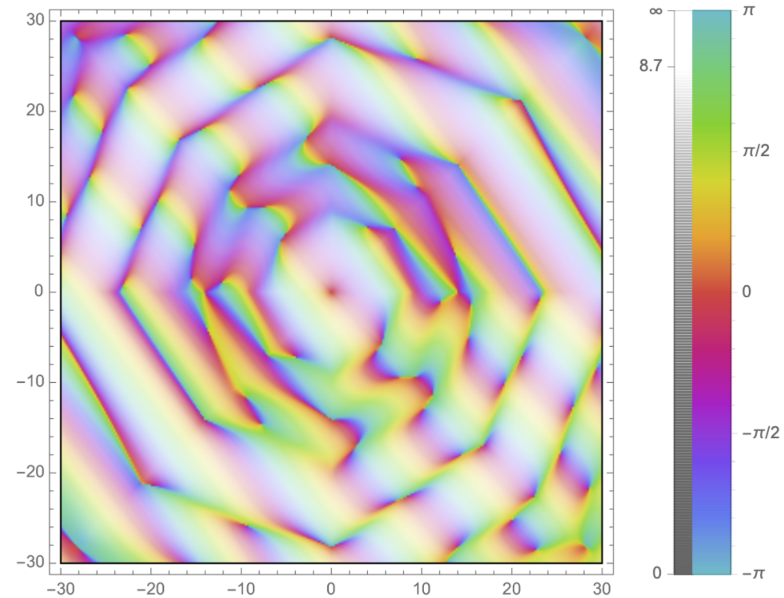
- **injectivity :** on cstr. fns. coming from persistence
- **regularity :** if S is a polytopal complex, continuous, piecewise-smooth, bounded
- **spectral** info on topology of slices

Software

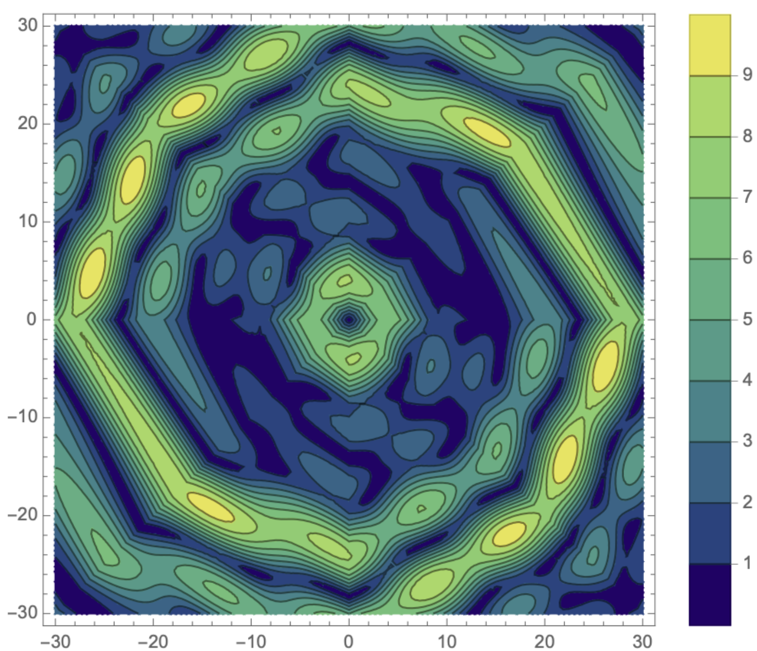
soon optimized in **C++** and **Python** by **Hugo Passe** (ENS Lyon)

Toy example : piecewise linear curve in $\mathbb{R}^2$

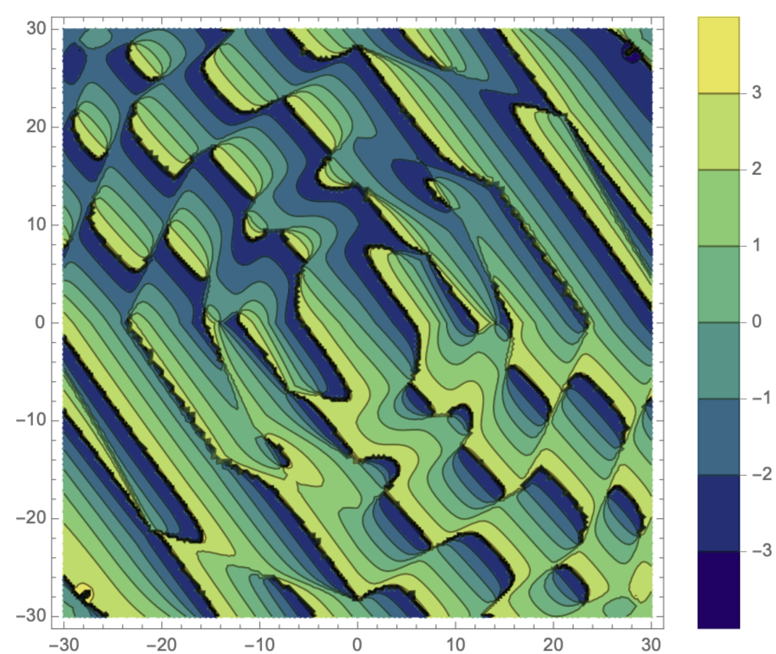




$\mathcal{EF}[L] : \mathbb{R}^2 \to \mathbb{C}$

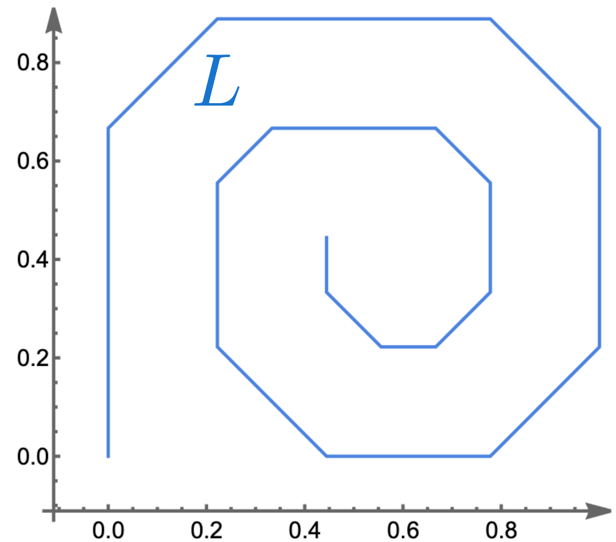


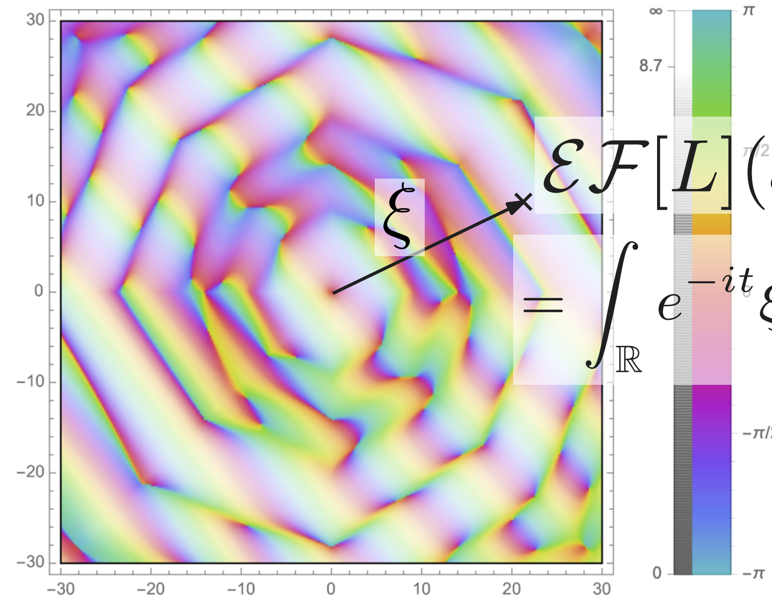
$|\mathcal{EF}[L]|$



$\mathrm{Arg}\,(\mathcal{EF}[L])$

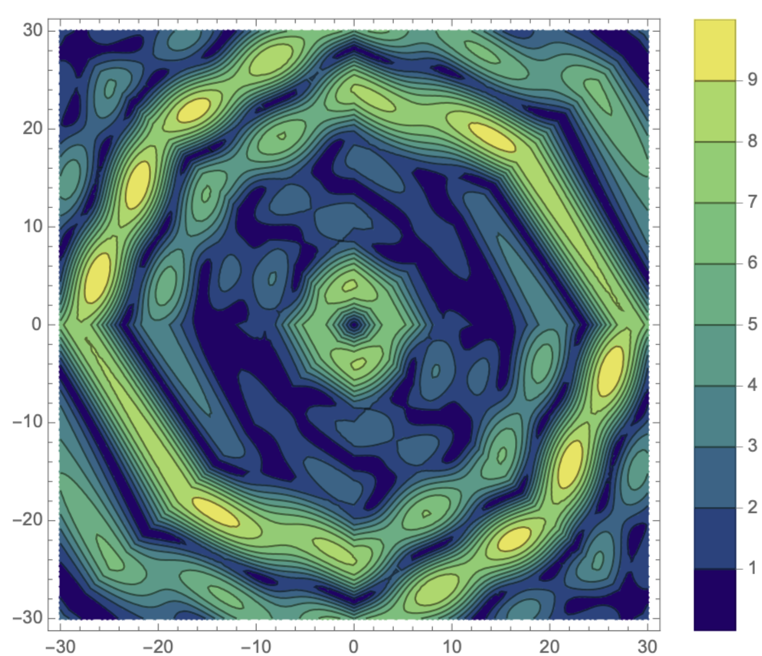
Toy example : piecewise linear curve in $\mathbb{R}^2$



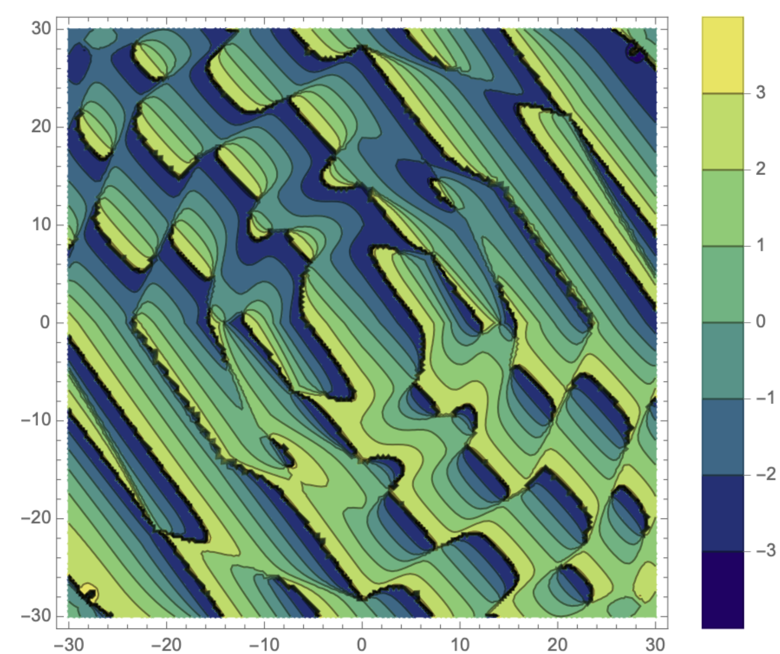


$$\mathcal{EF}[L](\xi) = \int_{\mathbb{R}} e^{-it\xi_*}\mathbf{1}_L(t)\,\mathrm{d}t$$

$\mathcal{EF}[L] : \mathbb{R}^2 \to \mathbb{C}$



$|\mathcal{EF}[L]|$



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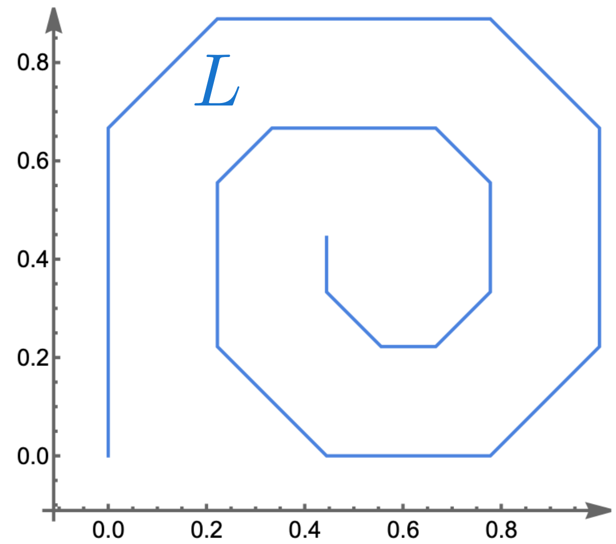
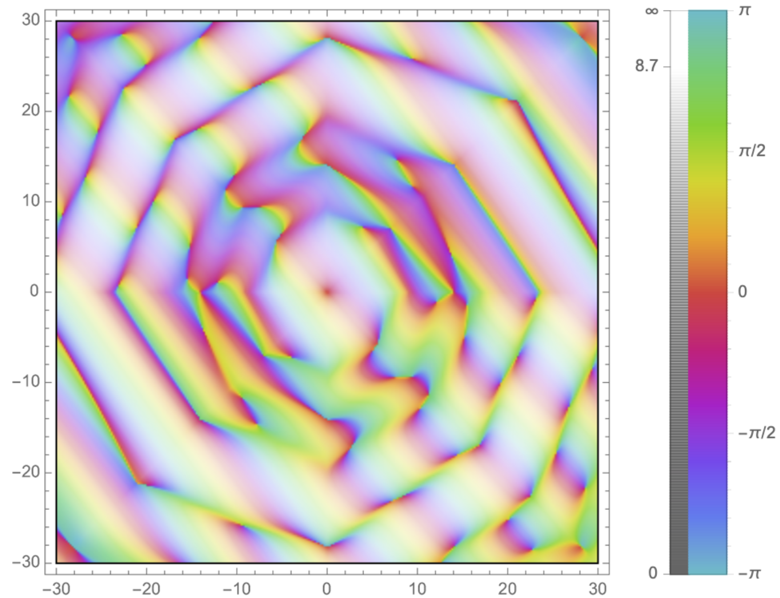
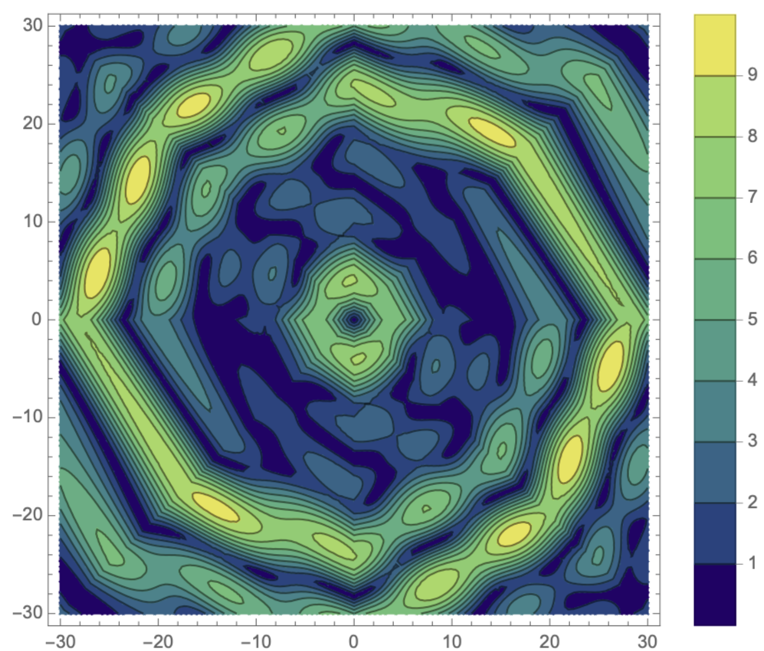
Toy example : piecewise linear curve in $\mathbb{R}^2$

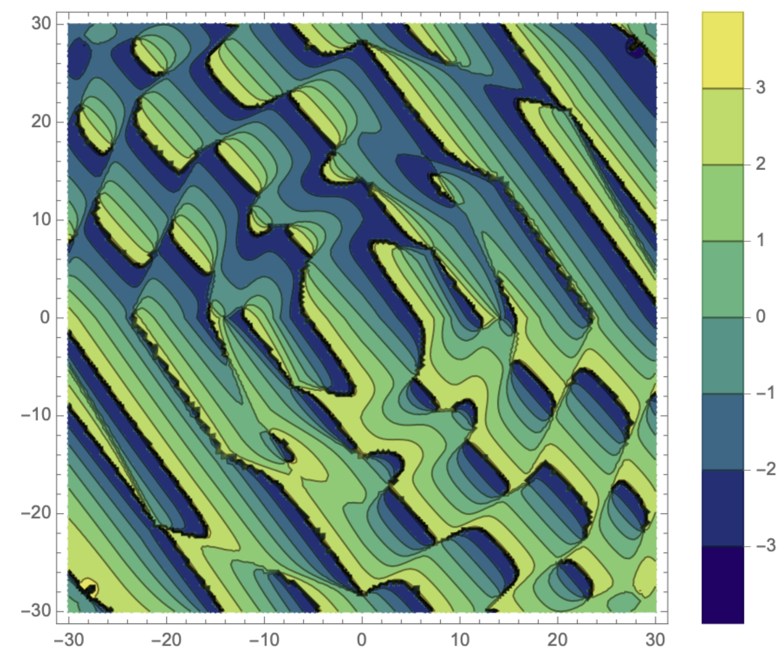




$\mathcal{E}\mathcal{F}[L] : \mathbb{R}^2 \to \mathbb{C}$

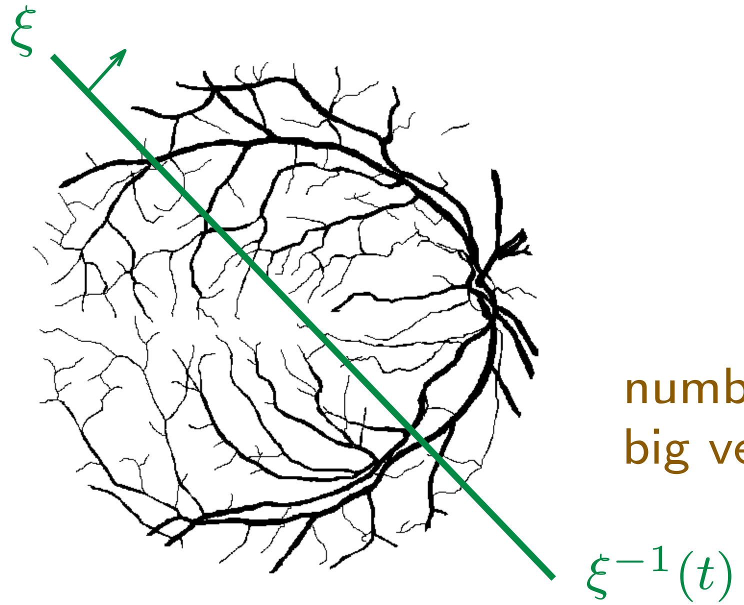


$|\mathcal{E}\mathcal{F}[L]|$

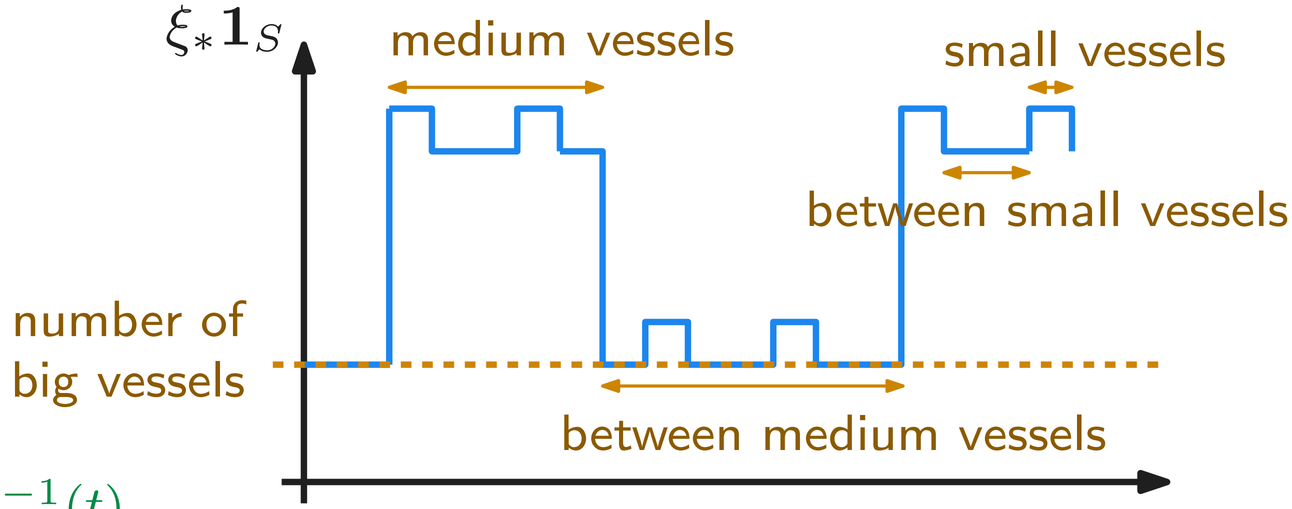


$\mathrm{Arg}\,(\mathcal{E}\mathcal{F}[L])$

Examples (2D)



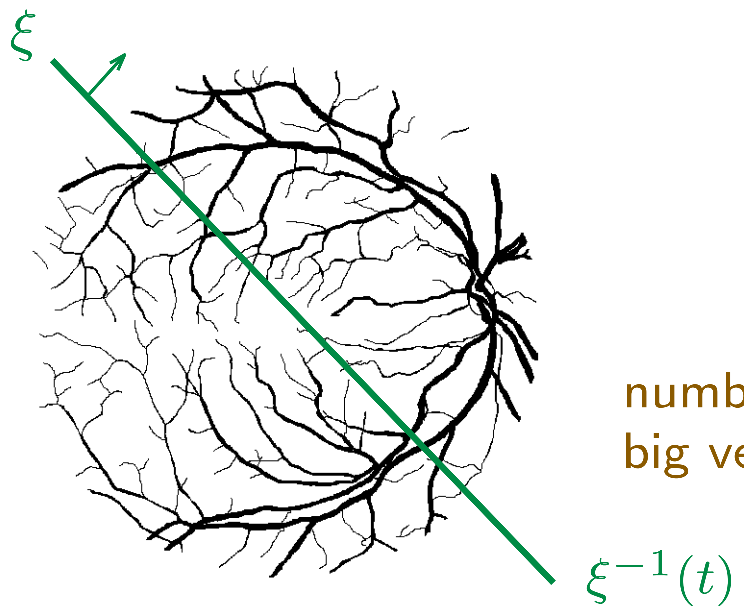
S = retina vessels [8]

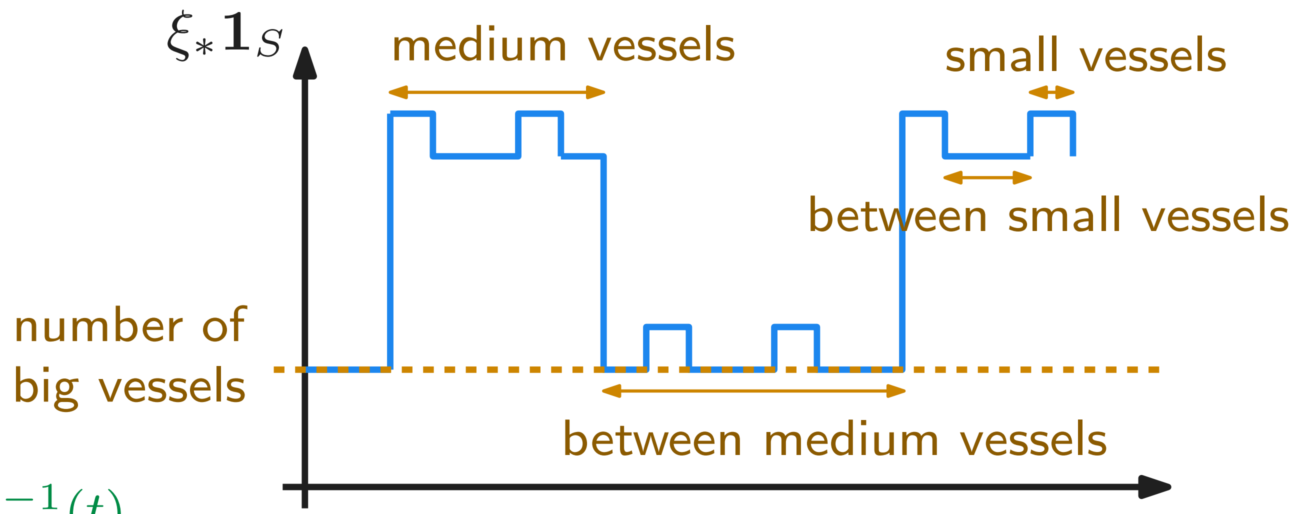


$\chi\big(S \cap \xi^{-1}(t)\big)$ = crossing number

[8] *DRIVE : Digital Retinal Images for Vessel Extraction*, `https://drive.grand-challenge.org`

Examples (2D)





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↓ Fourier

info | direction
size
frequence

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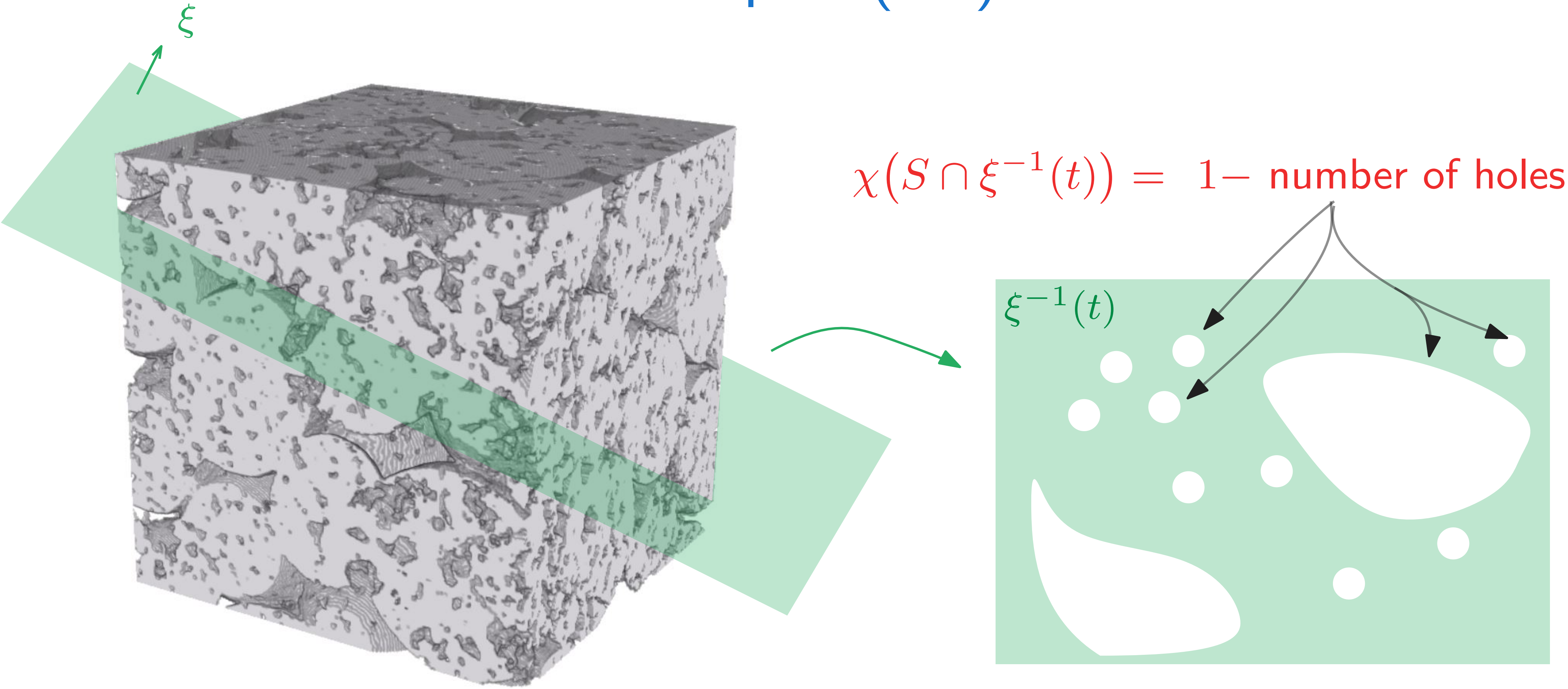
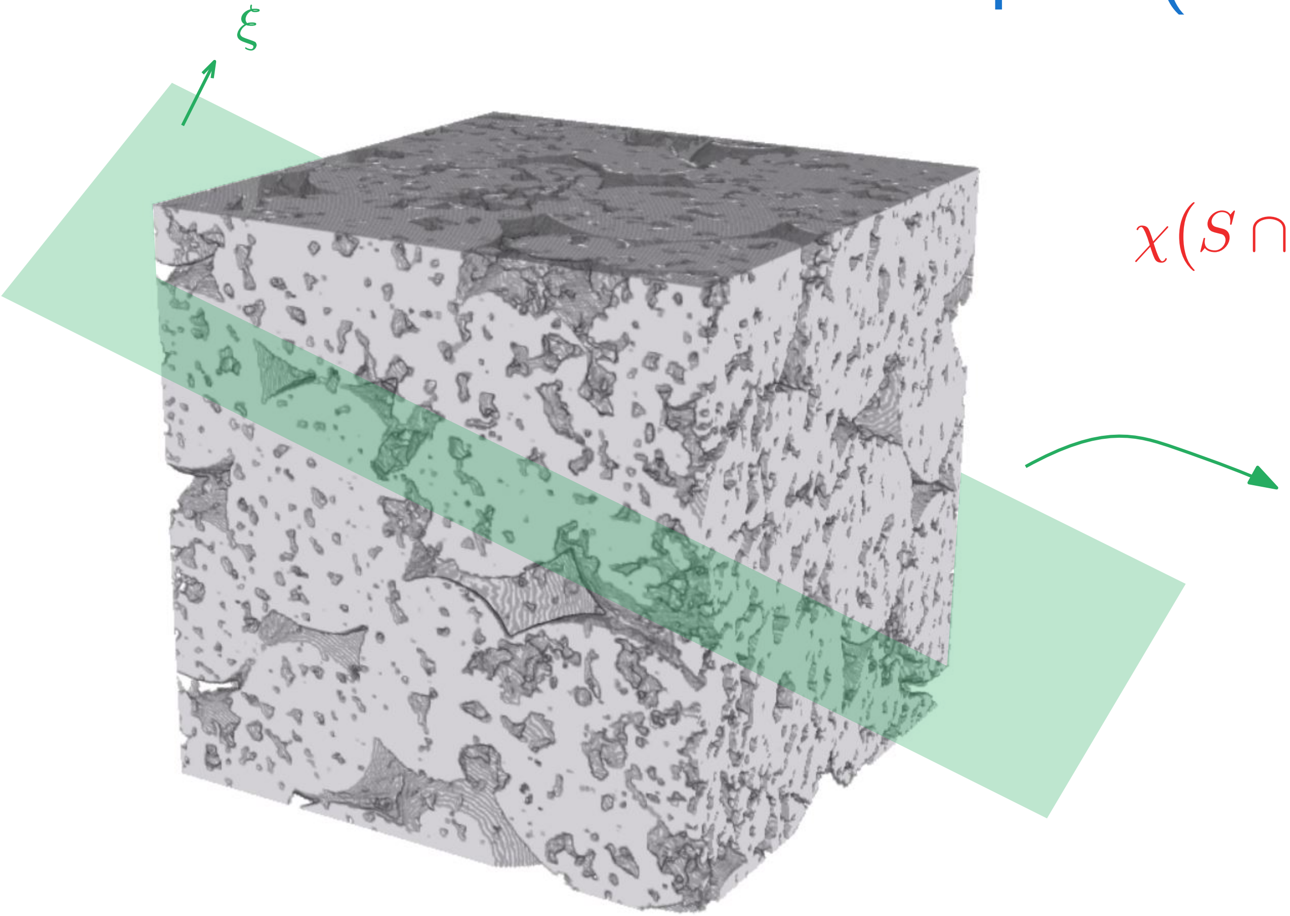
Examples (3D)



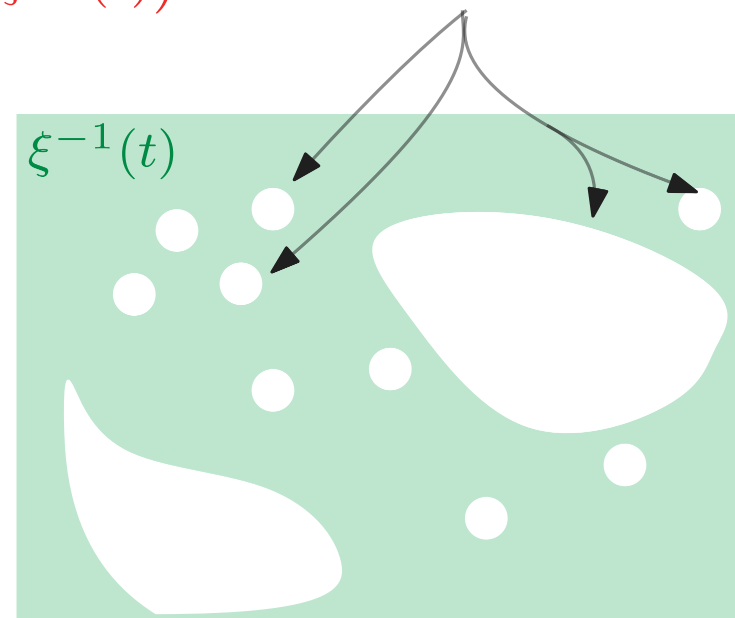
$S =$ sandy rock [7]

[7] Mohammadmoradi (2017) *A Multiscale Sandy Microstructure*, Digital Rocks Portal
`http://www.digitalrocksportal.org/projects/92`

Examples (3D)



$$\chi\big(S \cap \xi^{-1}(t)\big) = 1 - \text{number of holes}$$



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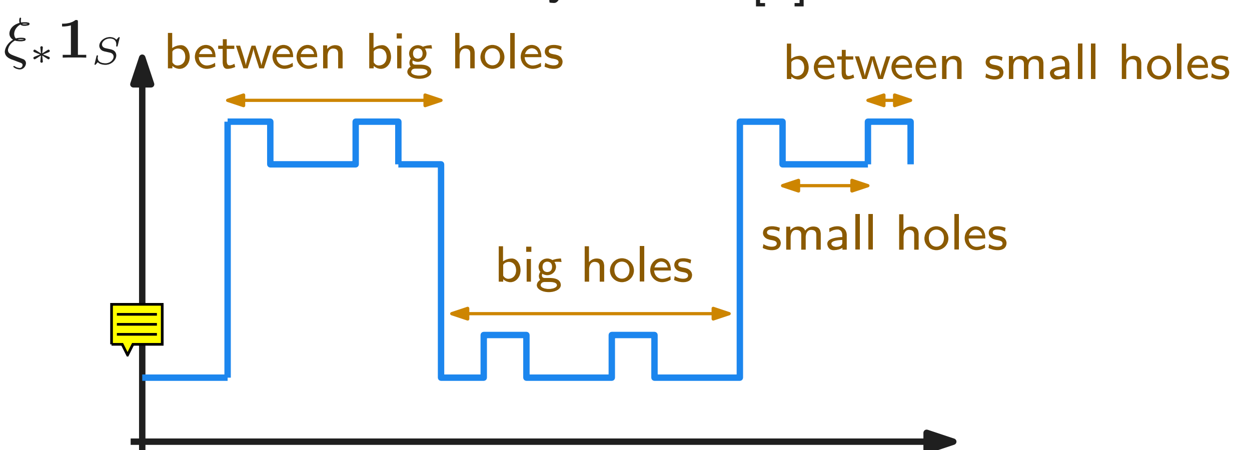


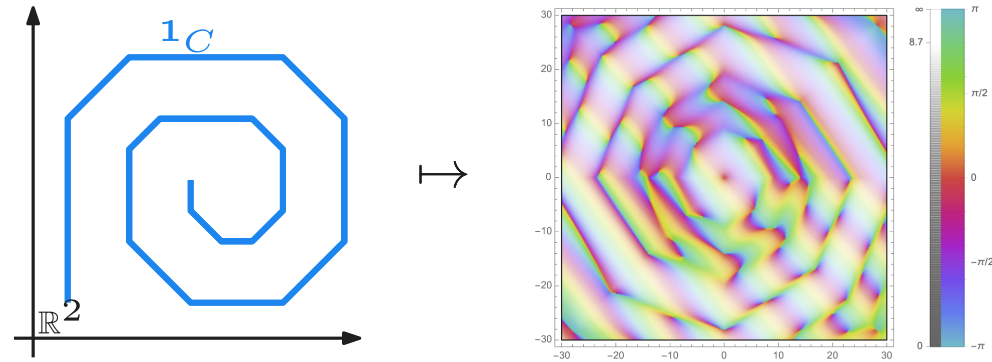
Fourier → info | direction, size, frequence

Conclusion

arXiv:2111.07829

Euler-Fourier transform

$$\mathcal{EF}[S] : \begin{array}{rcl} \mathbb{R}^n & \longrightarrow & \mathbb{C} \\ \xi & \longmapsto & \displaystyle\int_{\mathbb{R}} e^{-it}\xi_*\mathbf{1}_S(t)\,\mathrm{d}t \end{array}$$



Take-away : Fourier analysis of topological changes

Properties

- topological info
- well-suited to statistics
- interpretable
- continuous, piecewise-smooth (on polytopal complexes)

Conclusion

arXiv:2111.07829

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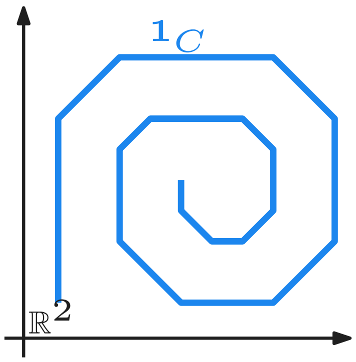
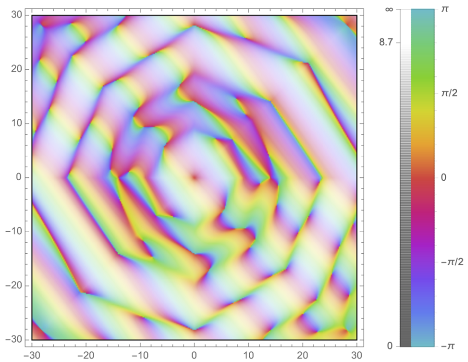


$\mapsto$



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Thank you !

References

[1] Schapira (1995) *Tomography of constructible functions.* In International Symposium on Applied Algebra, Algebraic Algorithms, and Error-Correcting Codes (pp. 427-435). Springer, Berlin, Heidelberg.

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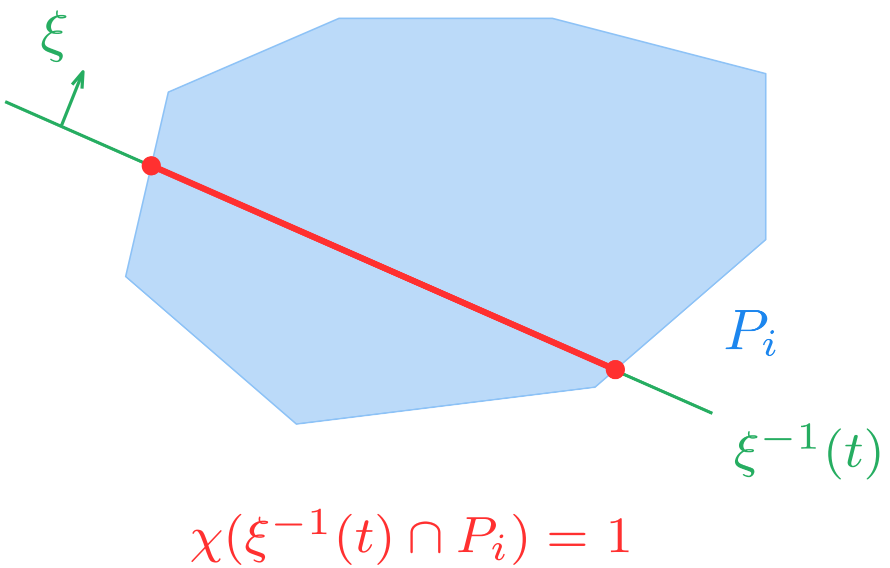
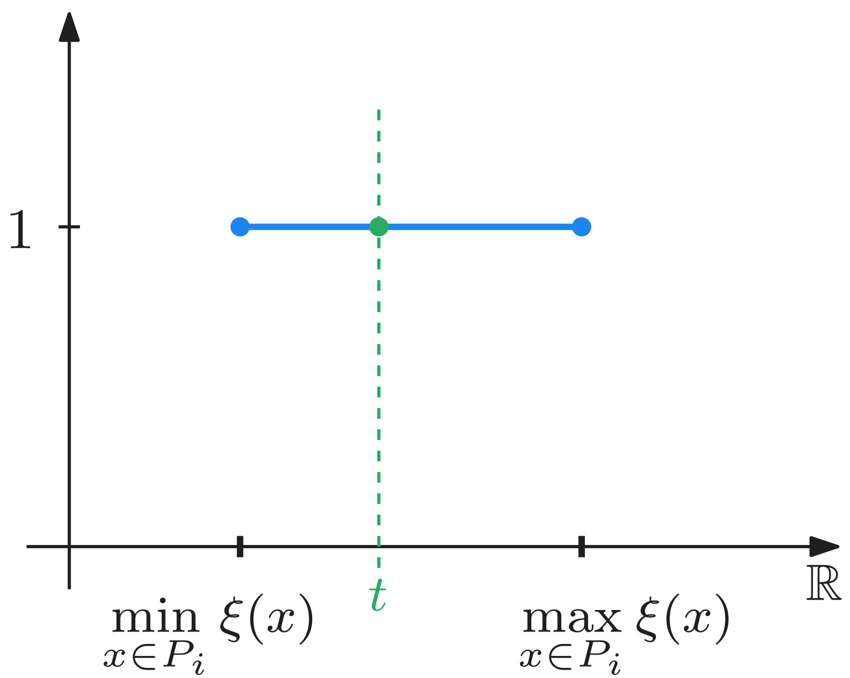
Computations

For $\varphi = \sum m_i \cdot \mathbf{1}_{P_i} \in \mathrm{CF}_{\mathrm{PL}}(\mathbb{R}^n)$, then $\mathcal{E}\mathcal{F}[\varphi] = \sum m_i \cdot \mathcal{E}\mathcal{F}[\mathbf{1}_{P_i}]$

Fact. $\xi_* \mathbf{1}_{P_i} = \mathbf{1}_{[\min_{P_i}(\xi), \max_{P_i}(\xi)]}$





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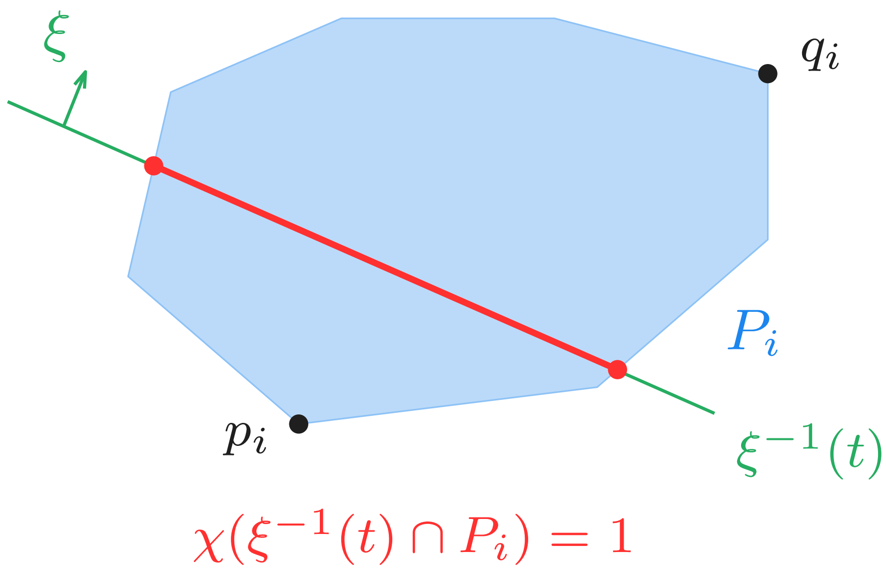

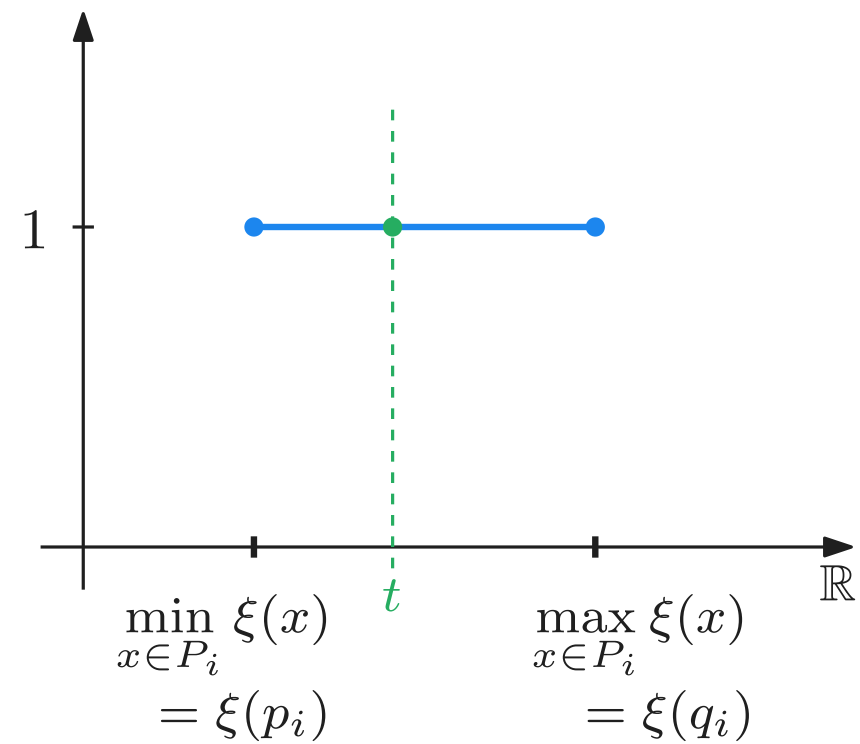

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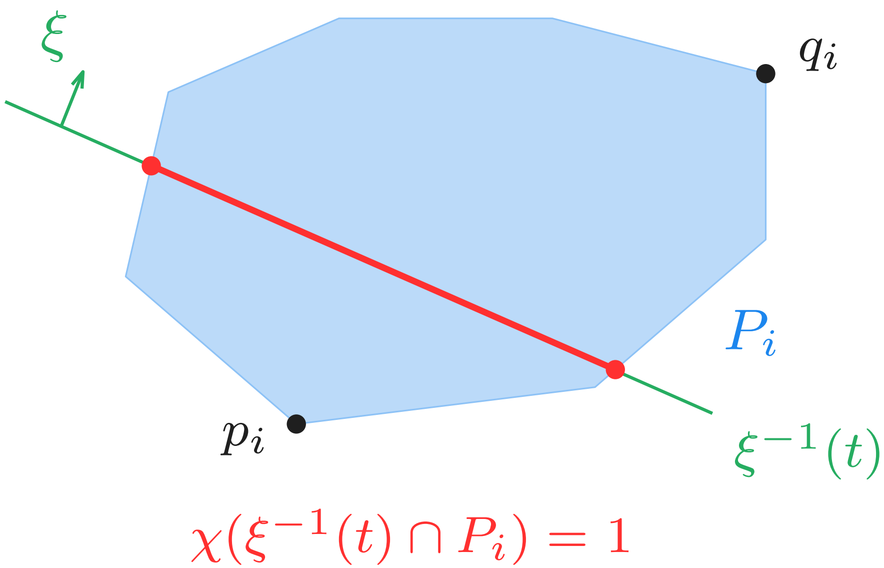
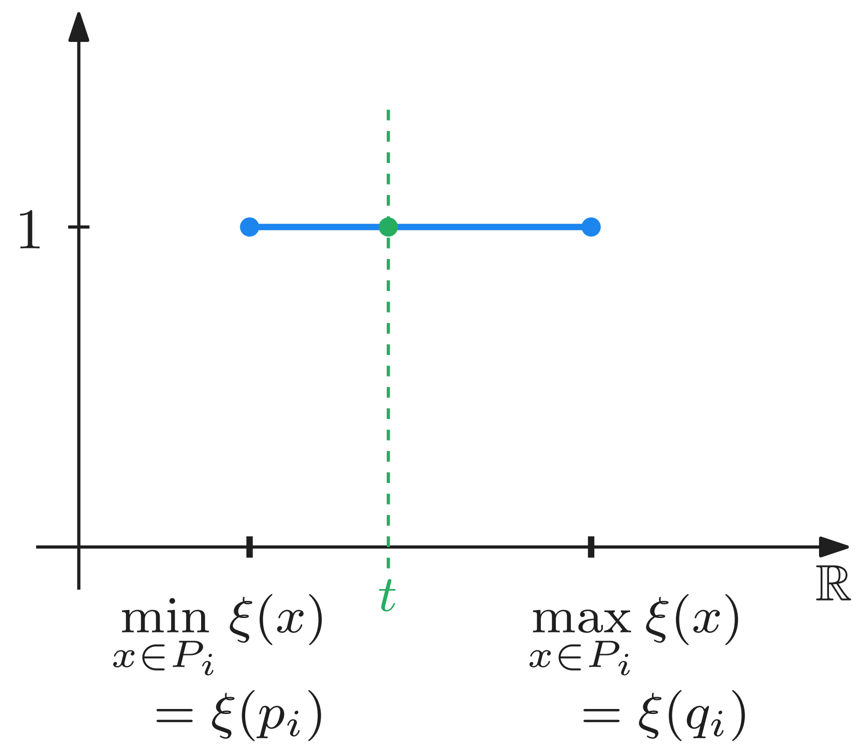


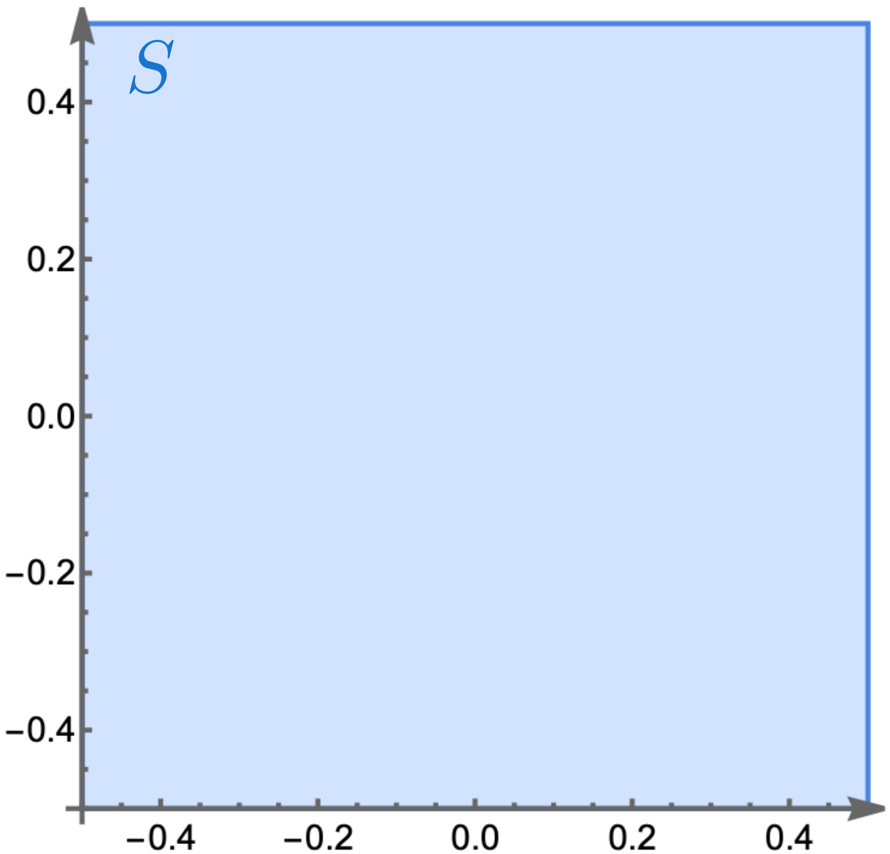


Computations.

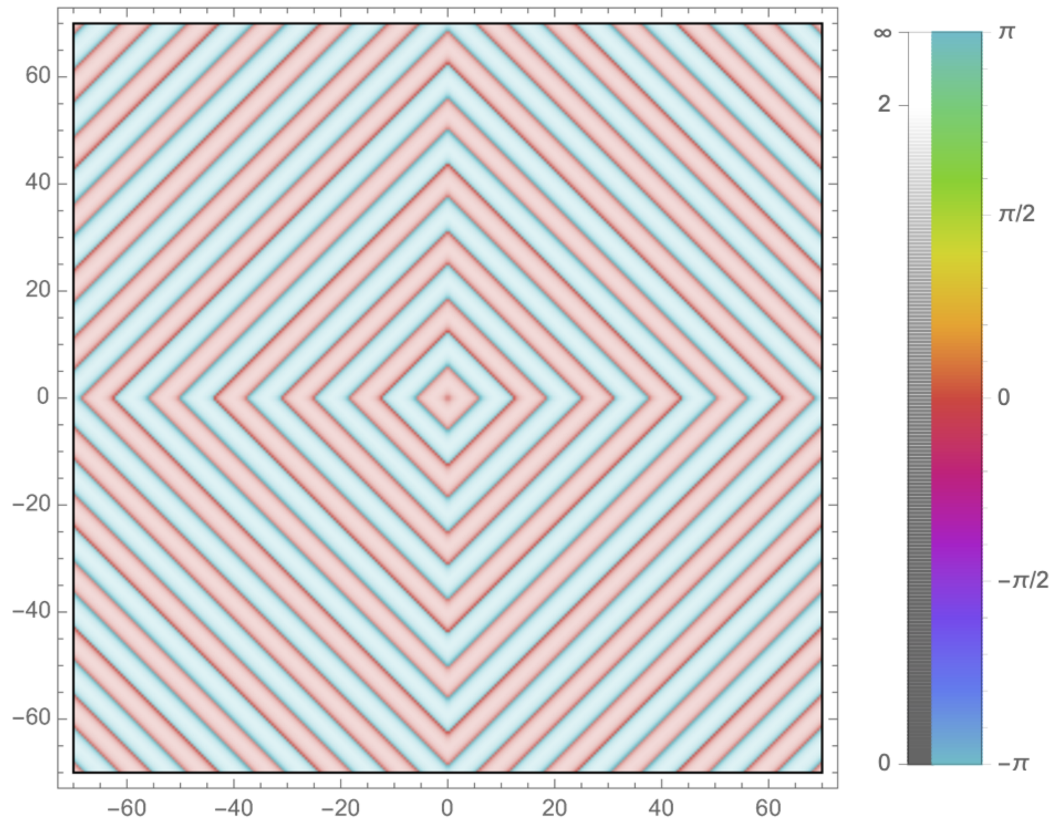
1. Compute $\min \xi$ and $\max \xi$ on $\mathrm{Vert}(P_i)$
2. Sum

$$\mathcal{E}\mathcal{F}[\varphi] = \sum m_i \int_{\mathbb{R}} e^{-it} \mathbf{1}_{[\xi(p_i), \xi(q_i)]} \, \mathrm{d}t = i \sum m_i \left(e^{-i\xi(p_i)} - e^{-i\xi(q_i)} \right)$$

Toy example : square

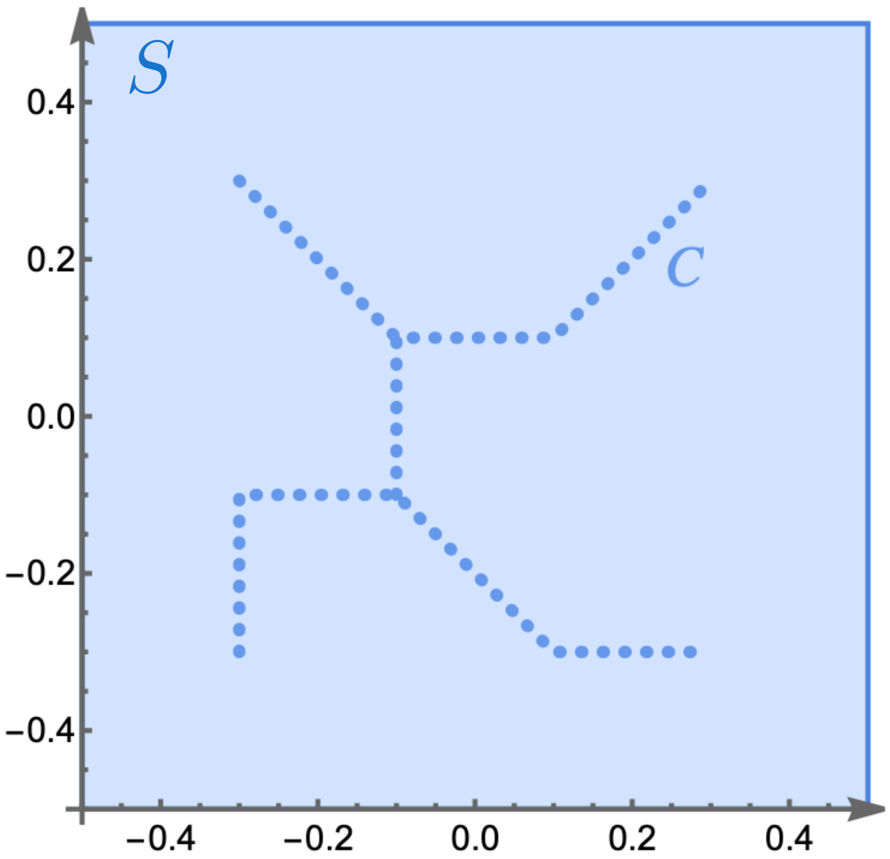


$\mathbf{1}_S$

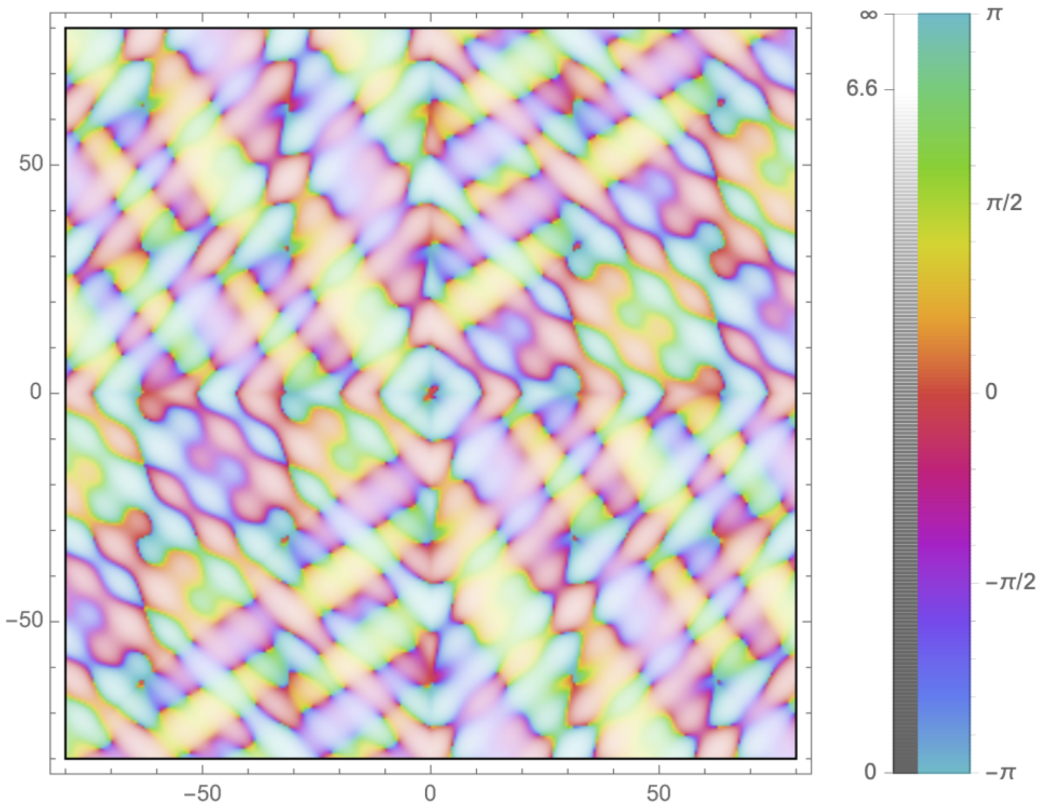


$\mathcal{E}\mathcal{F}[\mathbf{1}_S] : \mathbb{R}^2 \to \mathbb{C}$

Toy example : square minus a crack



$\mathbf{1}_S - \mathbf{1}_C$



$\mathcal{E}\mathcal{F}[\mathbf{1}_S - \mathbf{1}_C] : \mathbb{R}^2 \to \mathbb{C}$