Hybrid transforms of constructible functions

Vadim Lebovici

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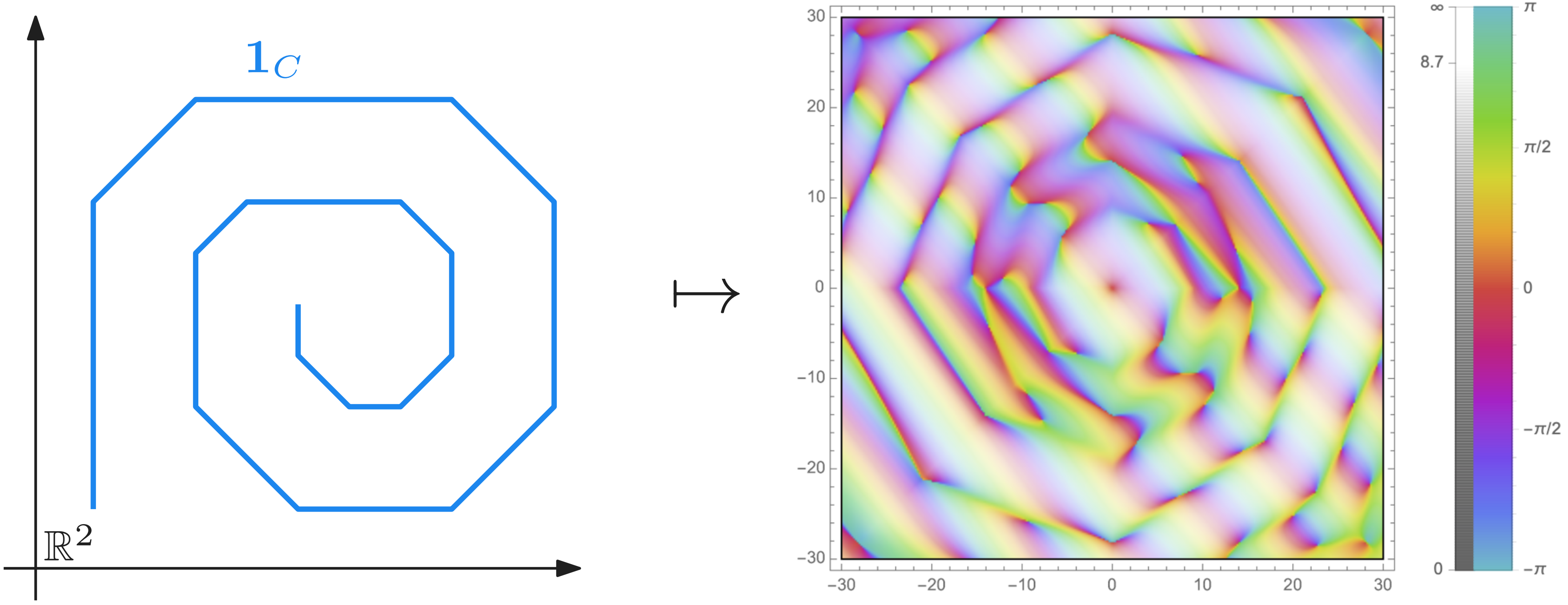


ATMCS 10 - 22/06/21

Overview

Integral transforms

$f : \mathbb{R}^n \to \mathbb{C}$ integrable $\xrightarrow{\text{Fourier}}$ $\underline{\mathcal{F}[f] : \mathbb{R}^n \to \mathbb{C}}$

records spectral info.

$\varphi : \mathbb{R}^n \to \mathbb{Z}$ constructible $\xrightarrow{\text{Radon}}$ $\underline{\mathcal{R}[\varphi] : \mathbb{R}^n \to \mathbb{Z}}$

records topological info.

$\varphi : \mathbb{R}^n \to \mathbb{Z}$ constructible $\xrightarrow{\text{Hybrid Fourier}}$ $\underline{\mathcal{E}\mathcal{F}[\varphi] : \mathbb{R}^n \to \mathbb{C}}$

records spectral info.
on topology

Overview

$$\varphi : \mathbb{R}^n \to \mathbb{Z} \ \text{ constructible} \quad \xrightarrow{\text{Hybrid transform}} \quad \mathcal{E}\mathcal{F}[\varphi] : \mathbb{R}^n \to \mathbb{C}$$

Main results

- Output function is **regular** (continuous, piecewise-smooth) on PL-functions
- **Compatible** with constructible operations
- **Generalize** known invariants (e.g. **persistent magnitude**)
- **Left-inversion theorem** for Euler-Fourier (under assumptions $\supseteq$ persistence)

Constructible functions

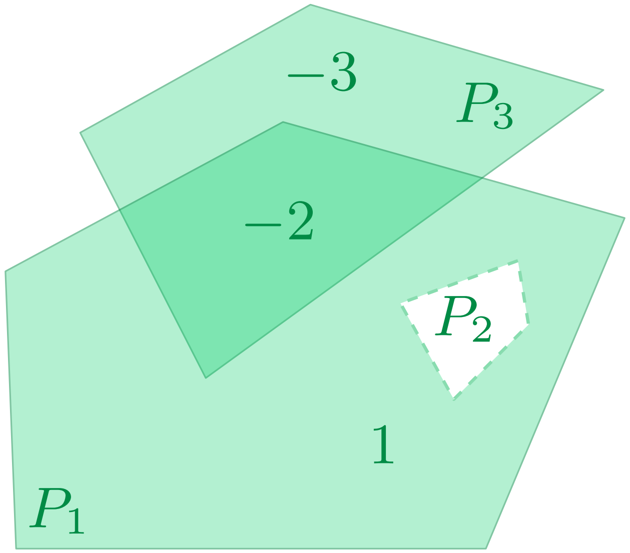
Def. (Constructible function)

$$\varphi = \sum_{i=1}^{k} m_i \mathbf{1}_{K_i}$$

where
- $m_i \in \mathbb{Z}$
- K_i compact subanalytic in $\mathbb{R}^n$ (e.g. polytope)

Not. $\varphi \in \mathrm{CF}(\mathbb{R}^n)$

Ex.



$$\varphi = \mathbf{1}_{P_1} - \mathbf{1}_{P_2} - 3 \cdot \mathbf{1}_{P_3} \in \mathrm{CF}_{\underline{\mathrm{PL}}}(\mathbb{R}^2)$$

Why constructible functions ?

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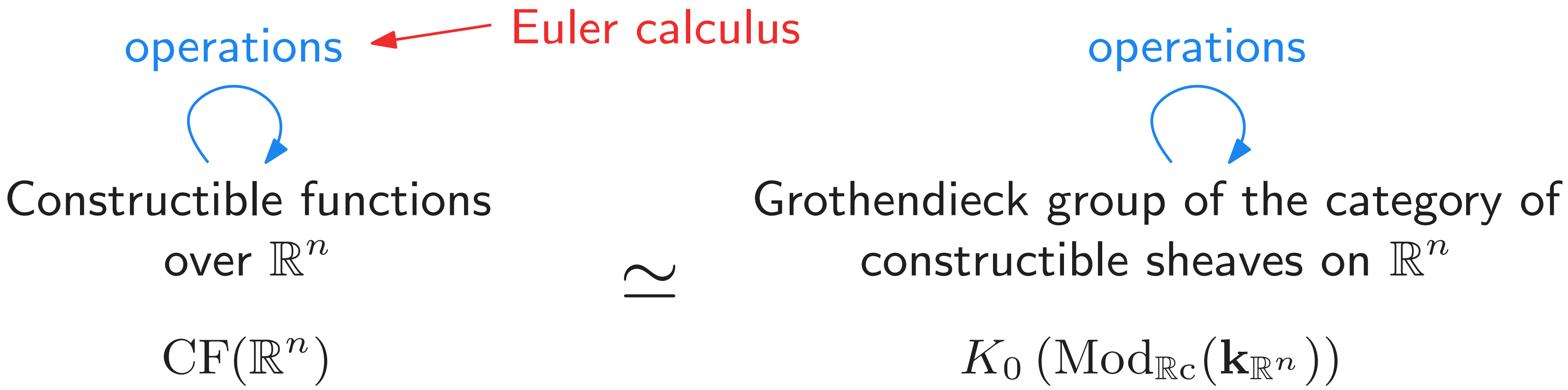
1. Theoretically rich Kashiwara, Schapira 1990



operations

Euler calculus

operations

Constructible functions over $\mathbb{R}^n$ $\simeq$ Grothendieck group of the category of constructible sheaves on $\mathbb{R}^n$

$\mathrm{CF}(\mathbb{R}^n)$ $\qquad$ $K_0\left(\mathrm{Mod}_{\mathbb{R}c}(\mathbf{k}_{\mathbb{R}^n})\right)$

Why constructible functions ?

1. **Theoretically rich**

2. **Useful in applied topology**

   (i) Persistence

pers. mod. on $\mathbb{R}^n$

$$M = \bigoplus_{j \in \mathbb{Z}} M_j \longmapsto \overbrace{\varphi_M : x \in \mathbb{R}^n \mapsto \sum_{j \in \mathbb{Z}} (-1)^j \dim M_j(x)}^{\text{invariant of pers. mod.}}$$

graded pers. mod. on $\mathbb{R}^n$

(+ constructibility assumptions)

$\in \mathrm{CF}(\mathbb{R}^n)$

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graded pers. mod. on $\mathbb{R}^n$

Ex. $(n = 1)$ For $f : X \to \mathbb{R}$ continuous subanalytic,

$$\varphi_f : \begin{array}{l} \mathbb{R} \to \mathbb{Z} \\ t \mapsto \chi\big(\{x \in X \,;\; f(x) \leq t\}\big) \end{array} \in \mathrm{CF}(\mathbb{R})$$

Here $M = \bigoplus_{j\in\mathbb{Z}} \mathrm{PH}_j(X, f)$

Why constructible functions ?

1. **Theoretically rich**
2. **Useful in applied topology**
   - **(i) Persistence**

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Ex. $(n = 1)$ For $f : X \to \mathbb{R}$ continuous subanalytic,

Euler curve →

$$\varphi_f : \left[\begin{array}{l} \mathbb{R} \to \mathbb{Z} \\ t \mapsto \chi\big(\{x \in X \ ; \ f(x) \leq t\}\big) \end{array}\right. \in \mathrm{CF}(\mathbb{R})$$

- Faster to compute

$$\chi(K) = \sum_{j\in\mathbb{Z}} (-1)^j \#\{j\text{-simplices}\}$$ if K simp. cplx

- Generalizes to $n \geq 1$ (multi-parameter persistence)

Why constructible functions ?

1. Theoretically rich

2. Useful in applied topology

pers. mod. on $\mathbb{R}^n$

invariant of pers. mod.

(i) Persistence $M = \bigoplus_{j\in\mathbb{Z}} M_j \longmapsto \overbrace{\varphi_M : x \in \mathbb{R}^n \mapsto \sum_{j\in\mathbb{Z}} (-1)^j \dim M_j(x)}$

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Too rough summary ?

- Faster to compute

- Generalizes to $n \geq 1$ (multi-parameter persistence)

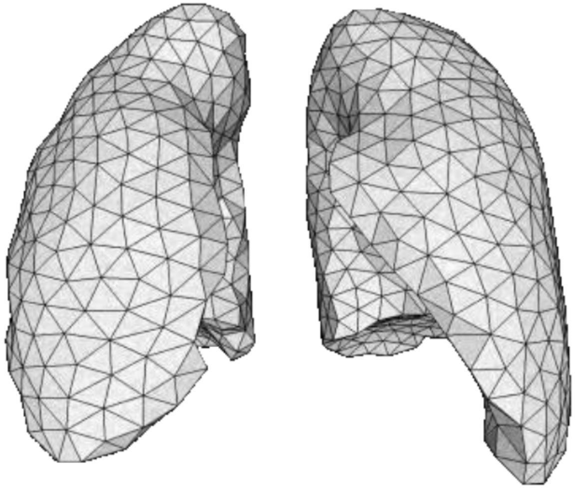
Why constructible functions ?

1. **Theoretically rich**
2. **Useful in applied topology**
    - **(i) Persistence**
    - **(ii) Topological integral transforms**

Why constructible functions ?

1. **Theoretically rich**
2. **Useful in applied topology**
   - **(i) Persistence**
   - **(ii) Topological integral transforms**
3. **Discrete object = constructible function**

   e.g. weighted simplicial/cubical complex



Euler calculus

Viro 1988
Schapira 1989

Let $\varphi = \sum_{i=1}^{k} m_i \mathbf{1}_{K_i} \in \mathrm{CF}(\mathbb{R}^n)$

compact subanalytic

Def. (Integration w.r.t. χ)

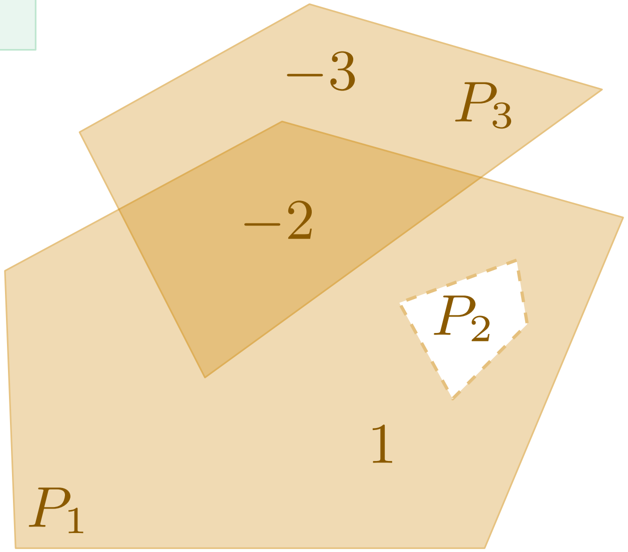
$$\int_{\mathbb{R}^n} \varphi \, \mathrm{d}\chi = \sum_{i=1}^{k} m_i \, \chi(K_i)$$

Ex. K compact subanalytic

$$\int_{\mathbb{R}^n} \mathbf{1}_K \, \mathrm{d}\chi = \chi(K)$$

Ex. $\varphi = \mathbf{1}_{P_1} - \mathbf{1}_{P_2} - 3 \cdot \mathbf{1}_{P_3} \in \mathrm{CF}_{\mathrm{PL}}\big(\mathbb{R}^2\big)$

$$\int_{\mathbb{R}^n} \varphi \, \mathrm{d}\chi = -3$$



Def. (Convolution) Let $\varphi, \psi \in \mathrm{CF}(\mathbb{R}^n)$.

$$\varphi \star \psi(x) = \int_{\mathbb{R}^n} \varphi(x-y)\psi(y) \, \mathrm{d}\chi(y)$$

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Def. (Pushforward) Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be analytic.

$$f_*\varphi : \begin{array}{ccc} \mathbb{R}^p & \longrightarrow & \mathbb{Z} \\ y & \longmapsto & f_*\varphi(y) = \displaystyle\int_{\mathbb{R}^n} \mathbf{1}_{f^{-1}(y)} \, \varphi \, \mathrm{d}\chi \end{array} \quad \in \mathrm{CF}(\mathbb{R}^p)$$

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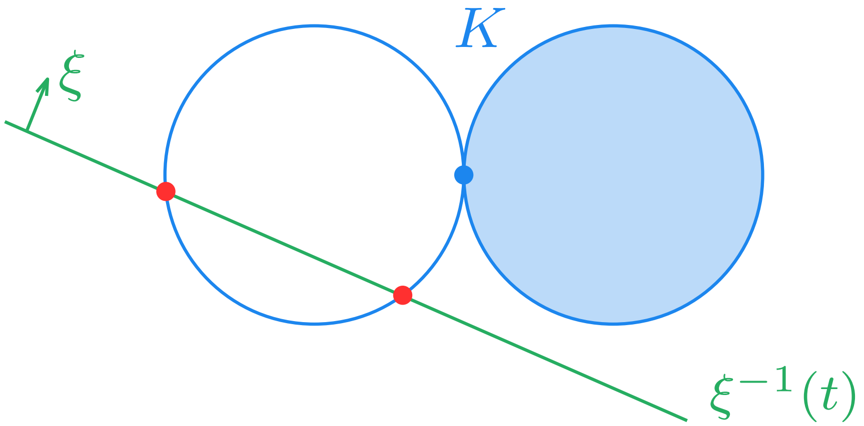
Ex. Consider $\xi : \mathbb{R}^n \to \mathbb{R}$ linear. $\varphi = \mathbf{1}_K \in \mathrm{CF}\big(\mathbb{R}^2\big)$



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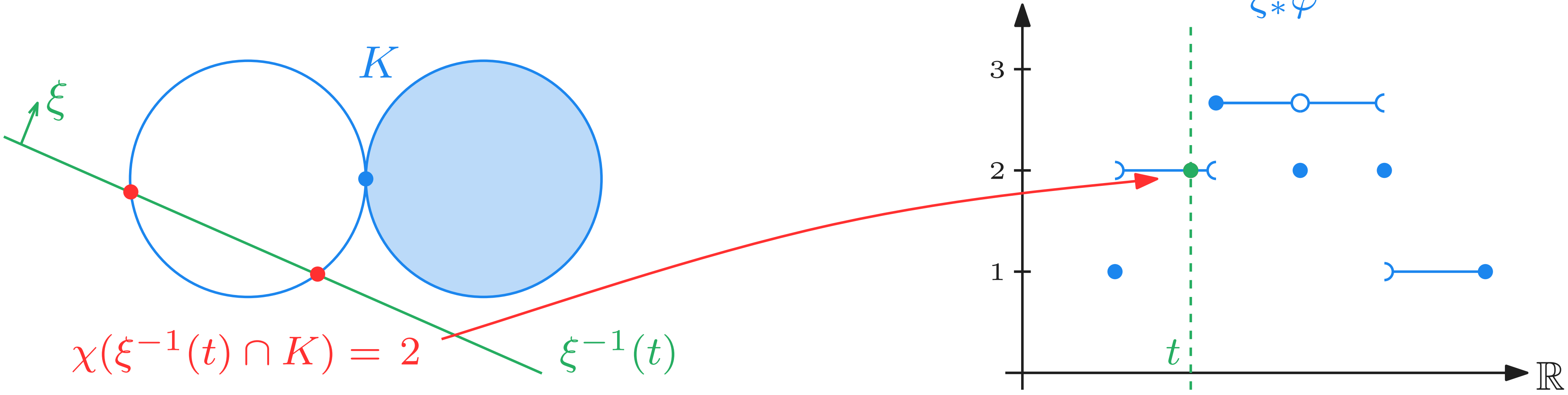

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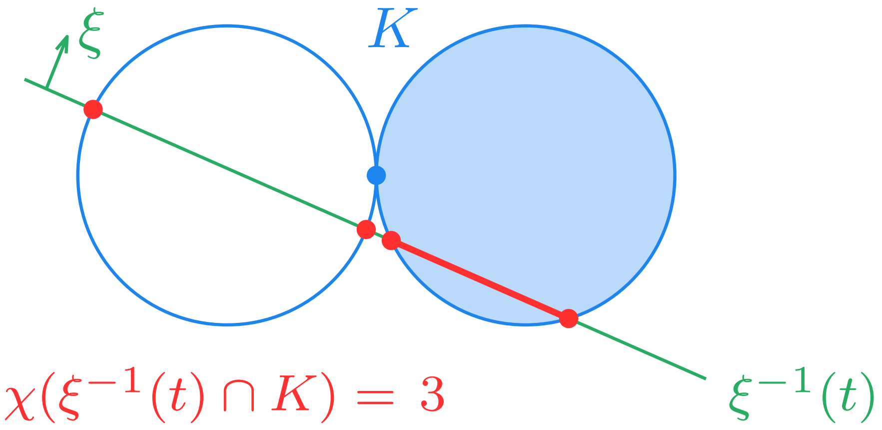

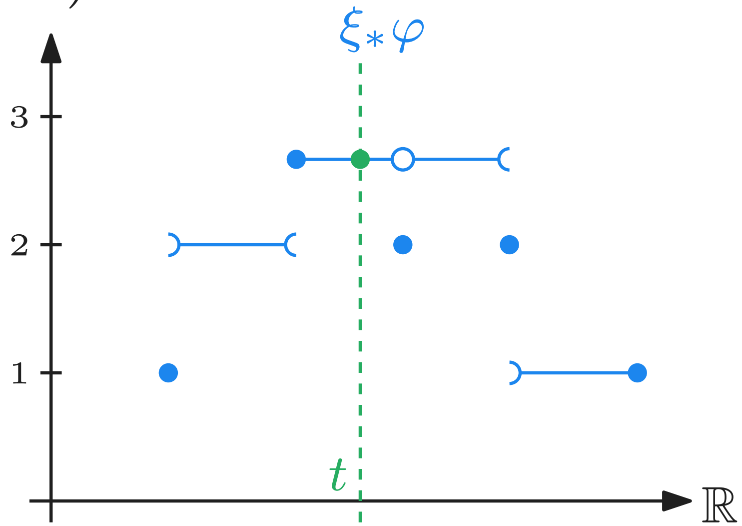

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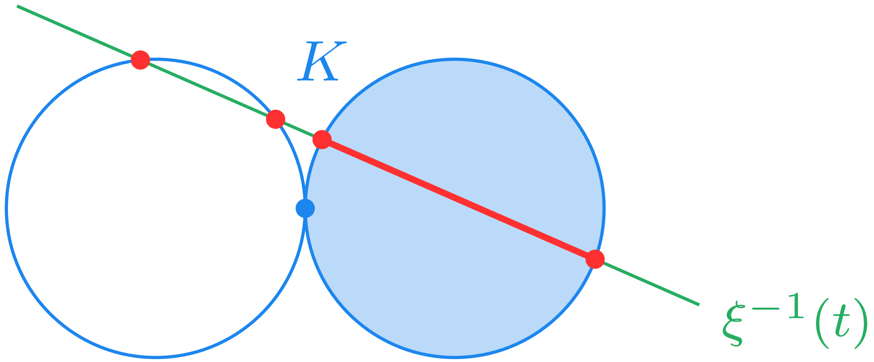
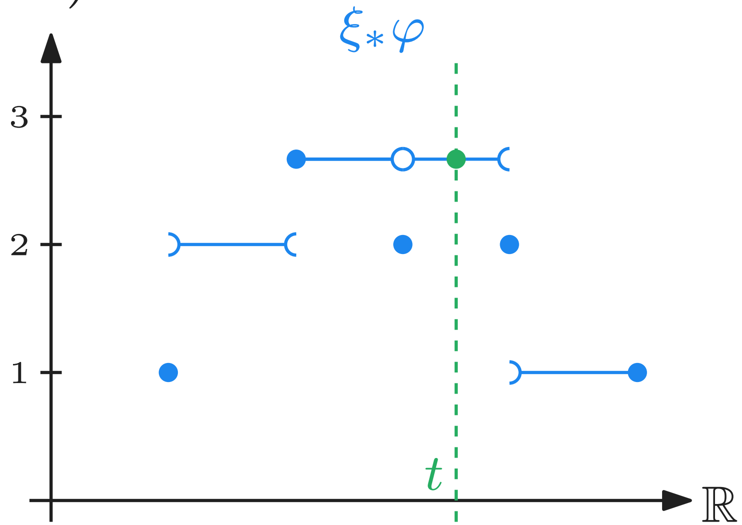


$\chi(\xi^{-1}(t) \cap K) = 3$



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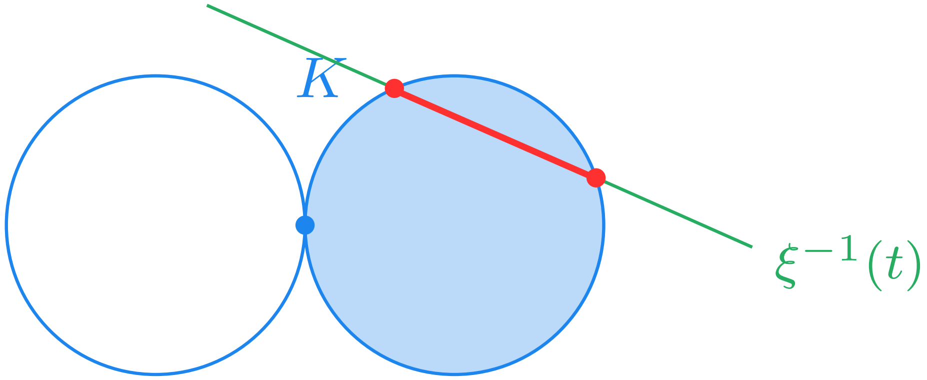
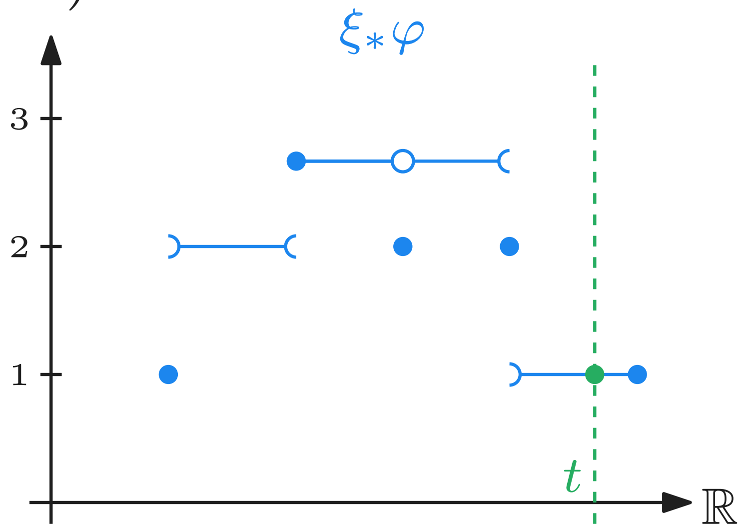


$\chi(\xi^{-1}(t) \cap K) = 1$



Topological integral transforms

Def. Radon transform Schapira 1995

$$\mathcal{R}[\varphi] : \begin{array}{ccc} \mathbb{S}^{n-1} \times \mathbb{R} & \longrightarrow & \mathbb{Z} \\ (\xi, t) & \longmapsto & \xi_* \varphi(t) \end{array}$$

Def. Euler characteristic transform (ECT) Turner, Mukherjee, Boyer '14

$$\mathrm{ECT}[\varphi] : \begin{array}{ccc} \mathbb{S}^{n-1} \times \mathbb{R} & \longrightarrow & \mathbb{Z} \\ (\xi, t) & \longmapsto & (\xi_* \varphi) \star \mathbf{1}_{[0,+\infty)} (t) \end{array}$$

Thm. Schapira [6]

$\mathcal{R} : \mathrm{CF}(\mathbb{R}^n) \to \mathrm{CF}\big(\mathbb{S}^{n-1} \times \mathbb{R}\big)$ is injective (up to a constant when n is even)

Thm. Turner, Mukherjee, Boyer '14, Curry, Mukherjee, Turner '18, Ghrist, Levanger, Mai '18

$\mathrm{ECT} : \mathrm{CF}(\mathbb{R}^n) \to \mathrm{CF}\big(\mathbb{S}^{n-1} \times \mathbb{R}\big)$ is injective.

Hybrid transforms

Def. (L. '21) (Hybrid transform) Let $\kappa : \mathbb{R} \to \mathbb{C}$ in L^1_{loc} and $\varphi \in \mathrm{CF}(\mathbb{R}^n)$.

$$\mathrm{T}_\kappa[\varphi] : \begin{array}{rcl} \mathbb{R}^n & \longrightarrow & \mathbb{C} \\ \xi & \longmapsto & \displaystyle\int_{\mathbb{R}} \kappa(t)\xi_*\varphi(t)\,\mathrm{d}t \end{array}$$

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Ex. Euler-Fourier

$$\mathcal{EF}[\varphi] : \begin{array}{rcl} \mathbb{R}^n & \longrightarrow & \mathbb{C} \\ \xi & \longmapsto & \displaystyle\int_{\mathbb{R}} e^{-it}\xi_*\varphi(t)\,\mathrm{d}t \end{array}$$

Ex. Euler-Laplace

$$\mathcal{EL}[\varphi] : \begin{array}{rcl} \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ \xi & \longmapsto & \displaystyle\int_{\mathbb{R}} e^{-t}\xi_*\varphi(t)\,\mathrm{d}t \end{array}$$

Multi-parameter
persistent magnitude

Hybrid transforms

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$$\mathrm{T}_\kappa[\varphi] : \begin{aligned} \mathbb{R}^n &\longrightarrow \mathbb{C} \\ \xi &\longmapsto \int_{\mathbb{R}} \kappa(t) \xi_* \varphi(t) \, \mathrm{d}t \end{aligned}$$

Generalizes

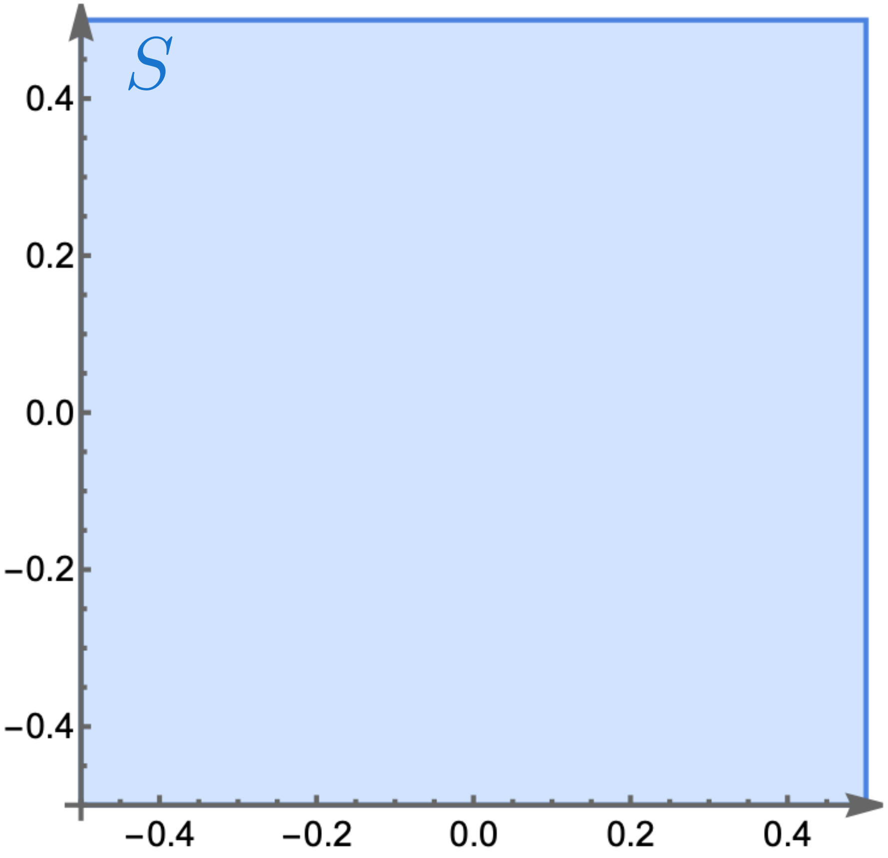
- "*Euler-Fourier*", *Euler-Bessel* in Ghrist, Robinson '11 (without kernel κ)
- *Persistent magnitude* in Govc, Hepworth '21 (on $\mathrm{CF}(\mathbb{R})$ with $\kappa(t) = e^{-t}$)

Related work

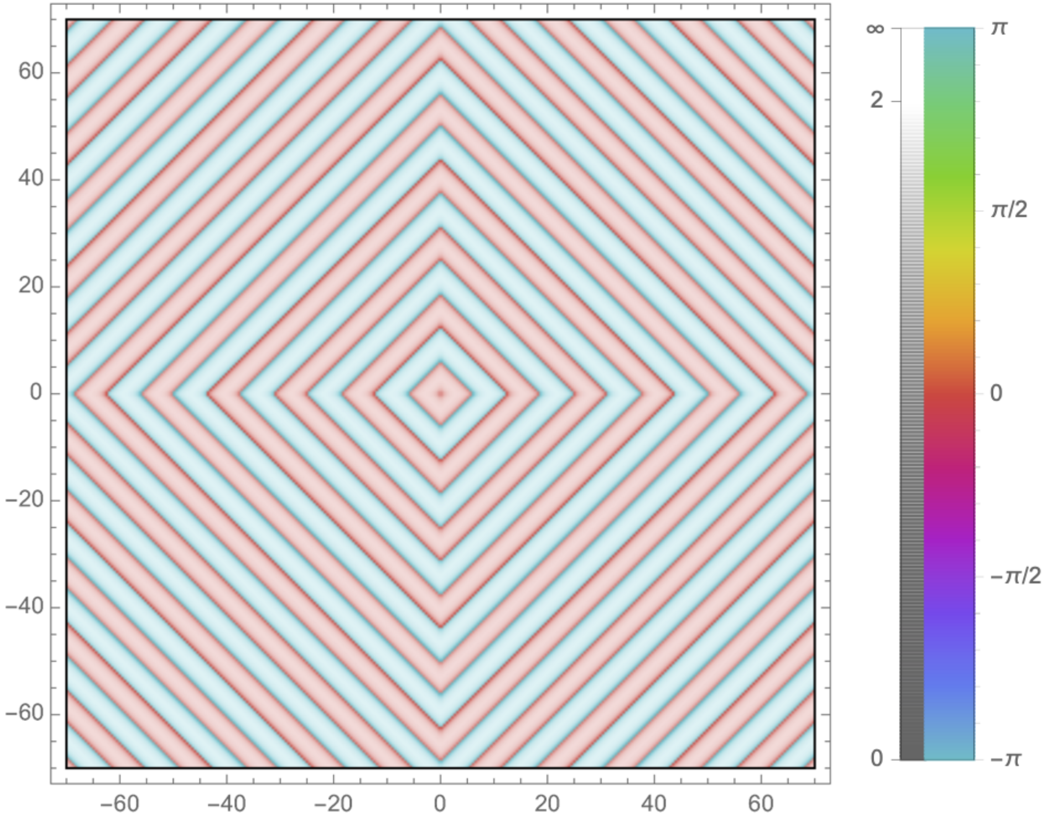
- *K-theoretic invariants* in Biran, Cornea, Zhang in *Persistence K-theory* (in prep.)

Example of hybrid transform

Square



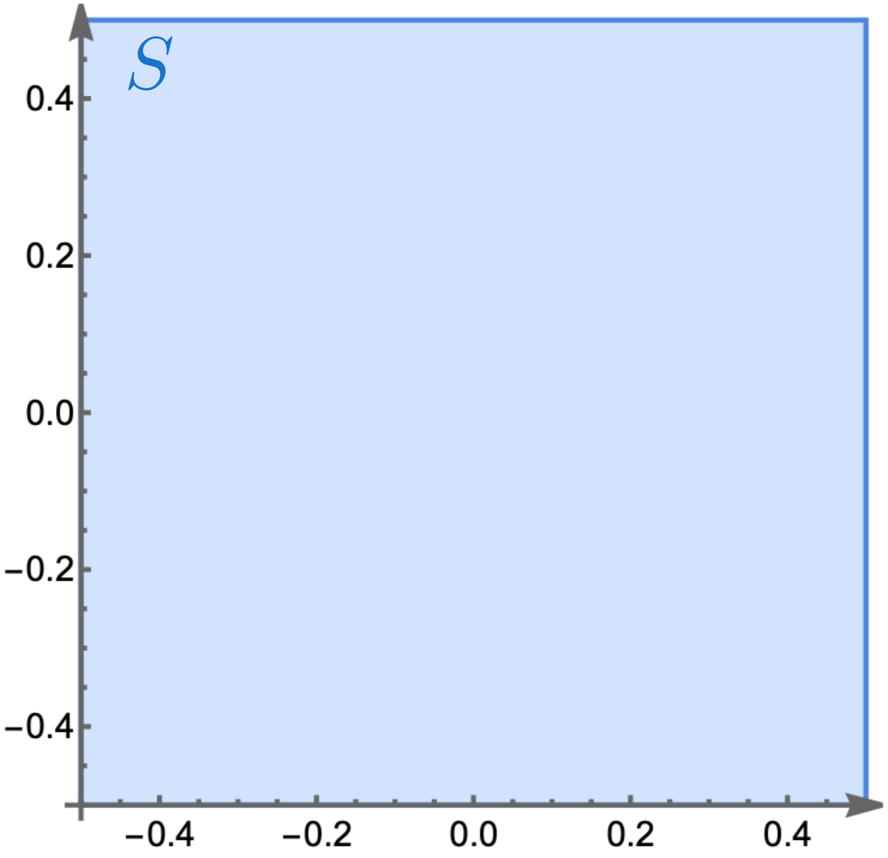
$$\mathbf{1}_S \in \mathrm{CF}\big(\mathbb{R}^2\big)$$



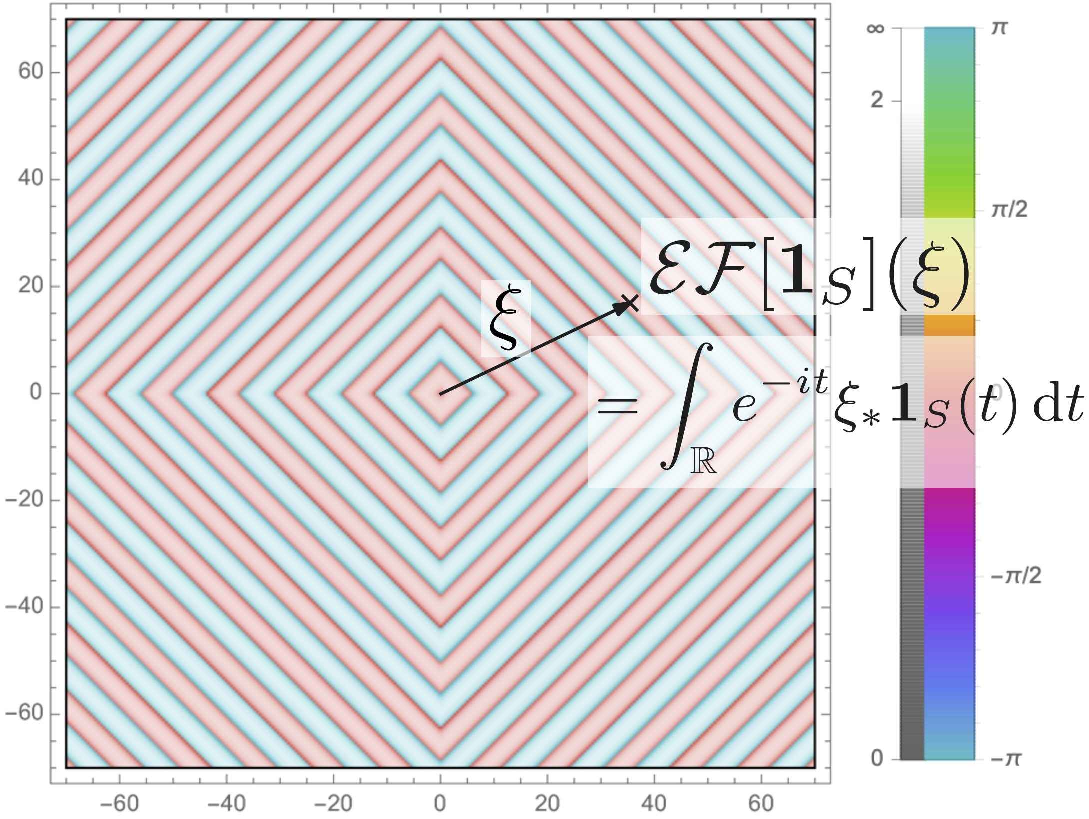
$$\mathcal{E}\mathcal{F}[\mathbf{1}_S] : \mathbb{R}^2 \to \mathbb{C}$$

Example of hybrid transform

Square



$$\mathbf{1}_S \in \mathrm{CF}(\mathbb{R}^2)$$

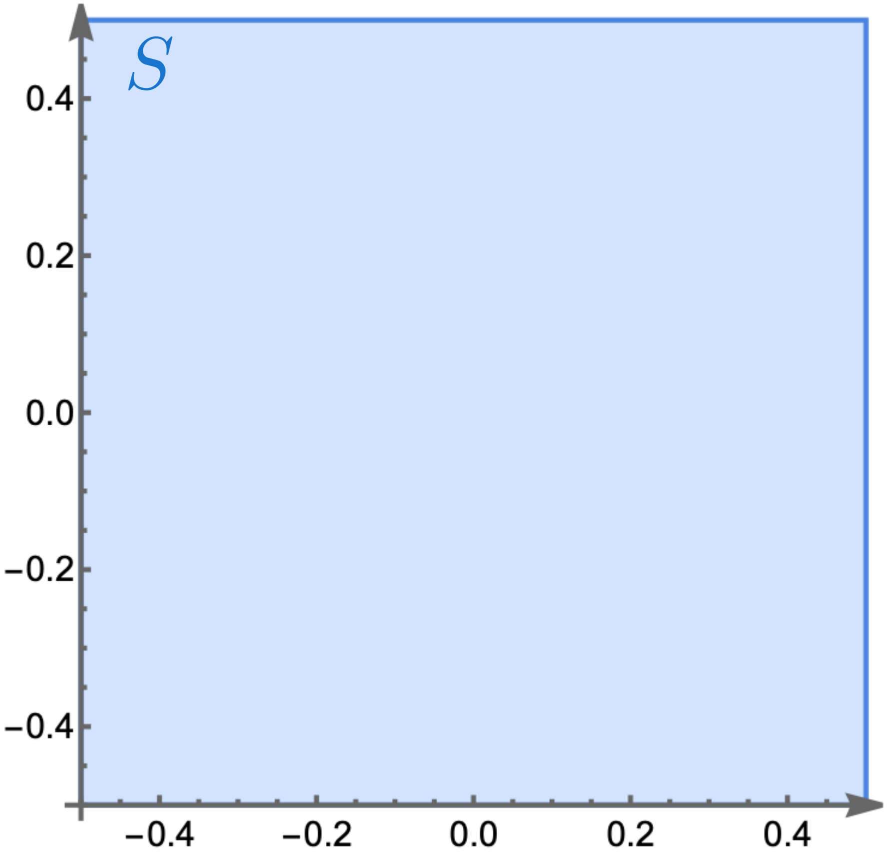


$$\mathcal{EF}[\mathbf{1}_S](\xi) = \int_{\mathbb{R}} e^{-it}\,\xi_*\mathbf{1}_S(t)\,\mathrm{d}t$$

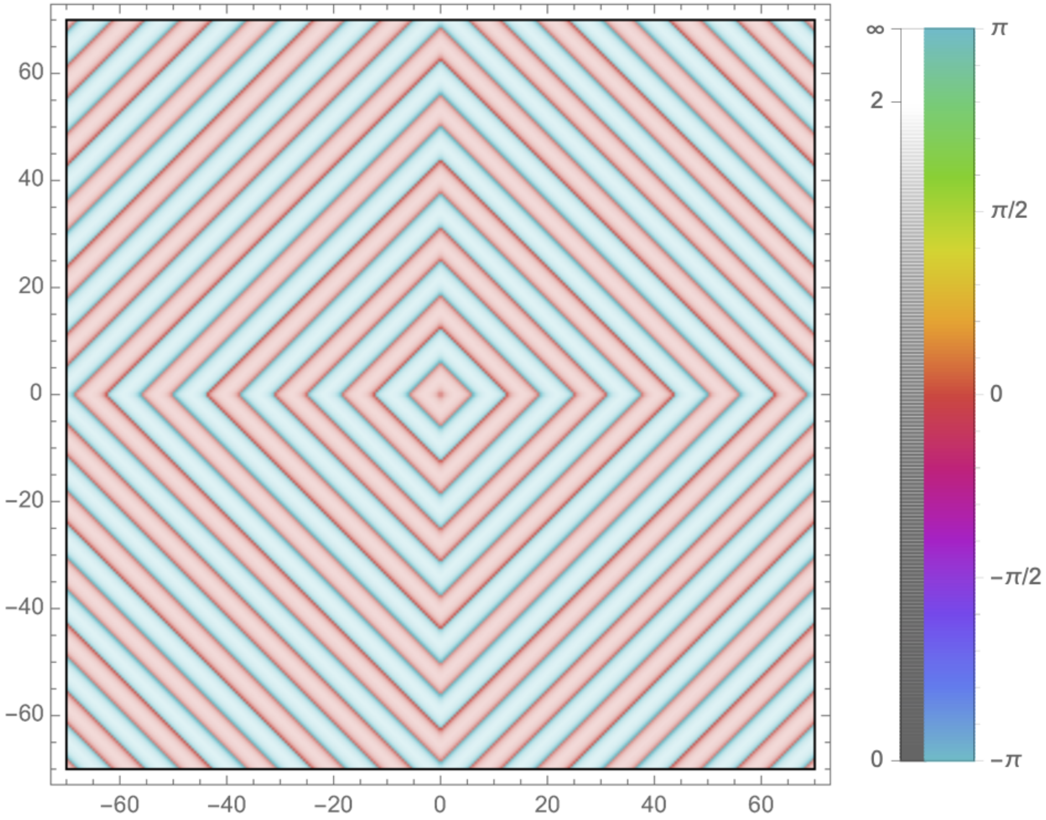
$$\mathcal{EF}[\mathbf{1}_S] : \mathbb{R}^2 \to \mathbb{C}$$

Example of hybrid transform

Square



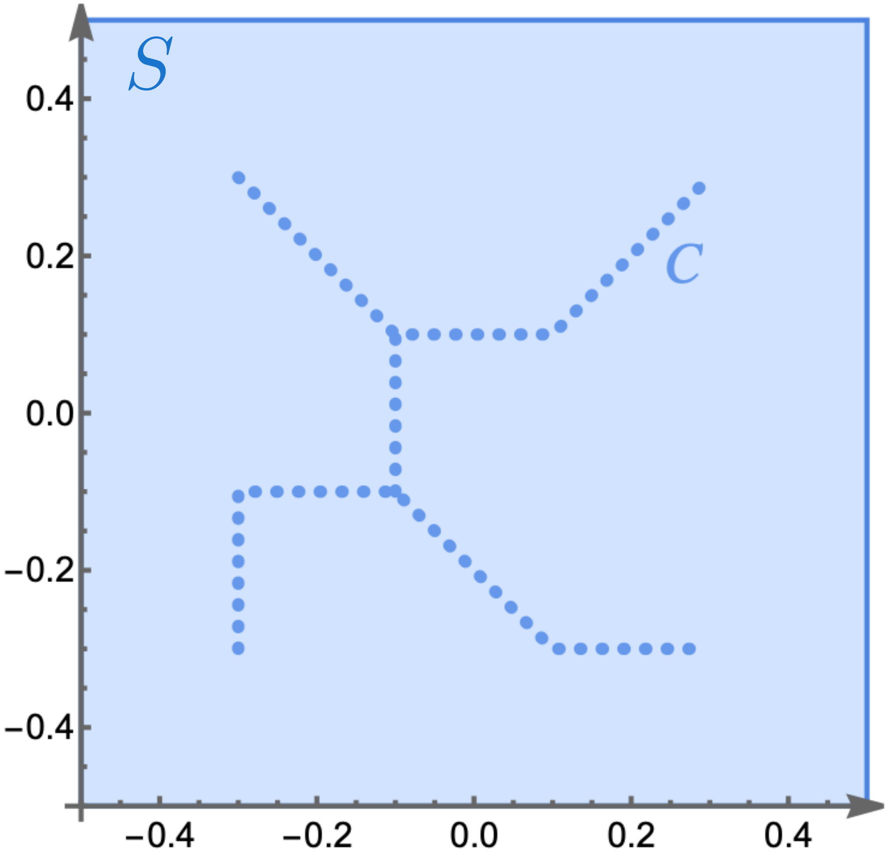
$\mathbf{1}_S \in \mathrm{CF}(\mathbb{R}^2)$



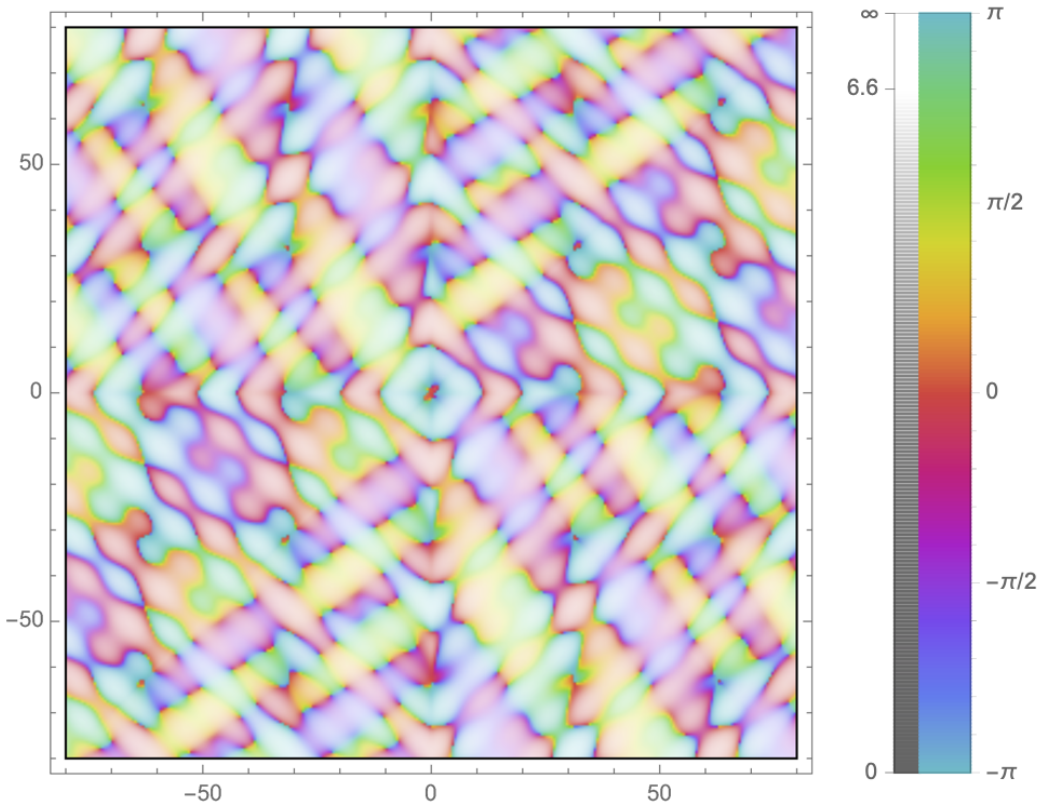
$\mathcal{E}\mathcal{F}[\mathbf{1}_S] : \mathbb{R}^2 \to \mathbb{C}$

Example of hybrid transform

Square minus a crack



$$\mathbf{1}_S - \mathbf{1}_C \in \mathrm{CF}\big(\mathbb{R}^2\big)$$

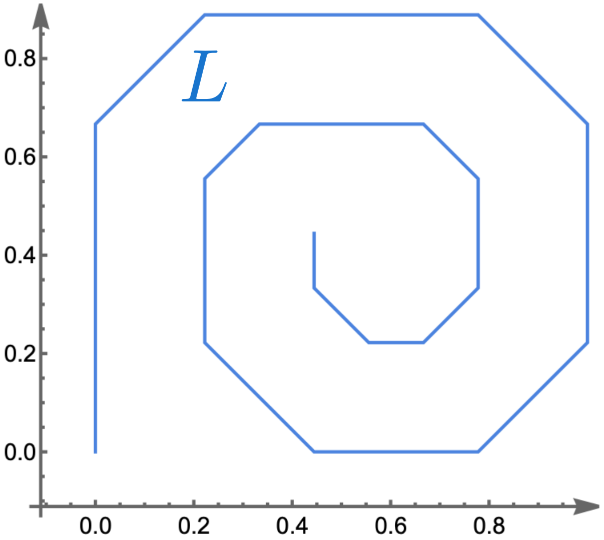


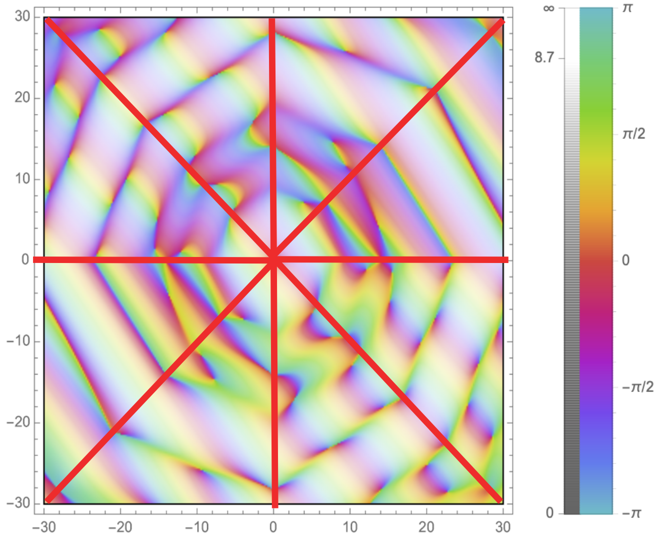
$$\mathcal{EF}[\mathbf{1}_S - \mathbf{1}_C] : \mathbb{R}^2 \to \mathbb{C}$$

Regularity

Prop. (L. '21) Let $\kappa : \mathbb{R} \to \mathbb{C}$ in L^1_{loc} and $\varphi \in \mathrm{CF}_{\underline{\mathrm{PL}}}(\mathbb{R}^n)$.

- $\mathrm{T}_\kappa[\varphi] : \mathbb{R}^n \to \mathbb{C}$ is continuous.
- If κ is C^p, then there exists open convex polyhedral cones $\{\Gamma_i\}_{i=1}^k$ s.t. :
  - $\mathbb{R}^n = \bigcup_{i=1}^{k} \overline{\Gamma}_i$
  - $\mathrm{T}_\kappa[\varphi] : \Gamma_i \to \mathbb{C}$ is C^{p+1}





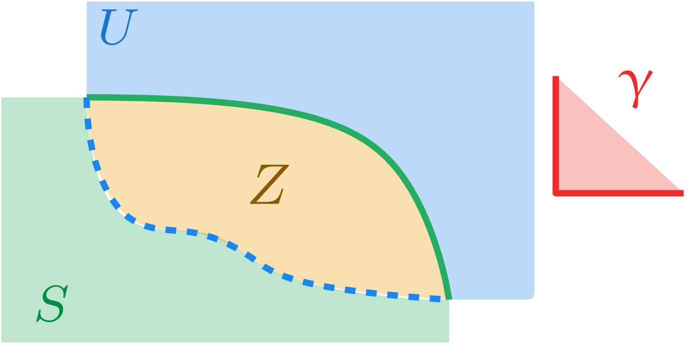
$\mathcal{E}\mathcal{F}[\mathbf{1}_L] : \mathbb{R}^2 \to \mathbb{C}$

Left-inversion of Euler-Fourier

Let $\gamma \subseteq \mathbb{R}^n$ be a closed convex subanalytic proper cone with non-empty interior.

Def. Z is γ-**locally closed** if $Z = U \cap S$

where
- U open s.t. $U + \gamma = U$
- S closed s.t. $S - \gamma = S$



Def. $\varphi \in \mathrm{CF}(\mathbb{R}^n)$ is γ-**constructible** if $\varphi = \sum_{i=1}^{k} m_i \mathbf{1}_{Z_i}$

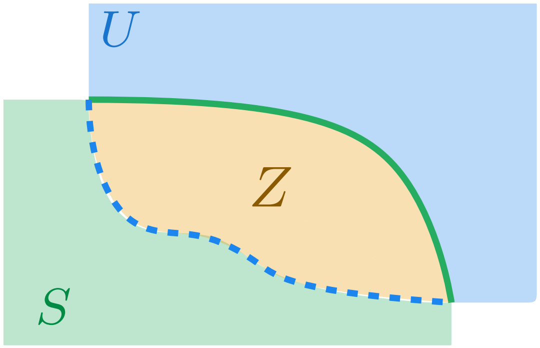
where $m_i \in \mathbb{Z}$,

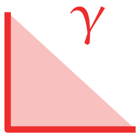
Z_i are γ-locally closed relatively compact.

Not. $\varphi \in \mathrm{CF}_\gamma(\mathbb{R}^n)$

Left-inversion of Euler-Fourier

Let $\gamma \subseteq \mathbb{R}^n$ be a closed convex subanalytic proper cone with non-empty interior.





Def. Z is γ-**locally closed** if $Z = U \cap S$

where
- U open s.t. $U + \gamma = U$
- S closed s.t. $S - \gamma = S$

Def. $\varphi \in \mathrm{CF}(\mathbb{R}^n)$ is γ-**constructible** if $\varphi = \sum_{i=1}^{k} m_i \mathbf{1}_{Z_i}$

where $m_i \in \mathbb{Z}$,

Z_i are γ-locally closed relatively compact.

Not. $\varphi \in \mathrm{CF}_\gamma(\mathbb{R}^n)$

Thm. (L. '21) $\mathcal{EF} : \mathrm{CF}_\gamma(\mathbb{R}^n) \to \mathcal{B}(\mathbb{R}^n, \mathbb{C})$ is injective.

bounded

Ex. $M = \bigoplus_{j \in \mathbb{Z}} M_j$ graded pers. mod. over $\mathbb{R}^n$

$$\varphi_M : x \in \mathbb{R}^n \mapsto \sum_{j \in \mathbb{Z}} (-1)^j \dim M_j(x) \quad \in \underline{\mathrm{CF}_\gamma(\mathbb{R}^n)}$$

Compatibility formulae

Prop. (L. '21) Let $\kappa \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $\varphi \in \mathrm{CF}(\mathbb{R}^n)$ and $f : \mathbb{R}^n \to \mathbb{R}^p$ analytic.

For any $\xi \in \mathbb{R}^p$, $$\mathrm{T}_\kappa[f_*\varphi](\xi) = \mathrm{T}_\kappa[\varphi](\xi \circ f)$$

Ex. Let $A \in \mathrm{GL}_n(\mathbb{R})$. Then $A_*\varphi(x) = \varphi(A^{-1}x)$, and

$$\mathrm{T}_\kappa[A_*\varphi](\xi) = \mathrm{T}_\kappa[\varphi]({}^tA\,\xi)$$

Prop. (L. '21) Let $\varphi, \psi \in \mathrm{CF}_\gamma(\mathbb{R}^n)$.

$$\mathcal{EL}[\varphi \star \psi] = \mathcal{EL}[\varphi] \cdot \mathcal{EL}[\psi]$$

↳ "Euler-Laplace turns constructible convolution into products."

and many others (e.g. with duality, ...)

Persistent magnitude Govc, Hepworth '21

Graded pers. mod. over $\mathbb{R}$ ⟶ **Persistent magnitude function**

$$M = \bigoplus_{j \in \mathbb{Z}} M_j$$

$$M_j \simeq \bigoplus_k \mathbf{k}_{[a_k^j, b_k^j)}$$

(finitely presented)

$$|M| : \begin{array}{rl} \mathbb{R}_{>0} & \to \mathbb{R} \\ t & \mapsto \sum_{j,k} (-1)^j \left(e^{-ta_k^j} - e^{-tb_k^j} \right) \end{array}$$

Prop. (Index theoretic formula) If $f : X \to \mathbb{R}$ is Morse, then

$$|\mathrm{PH}(X, f)|(t) = \sum_{p \in \mathrm{Crit(f)}} (-1)^{\mu(p)} e^{-tf(p)}$$

↗ sublevel-sets persistence

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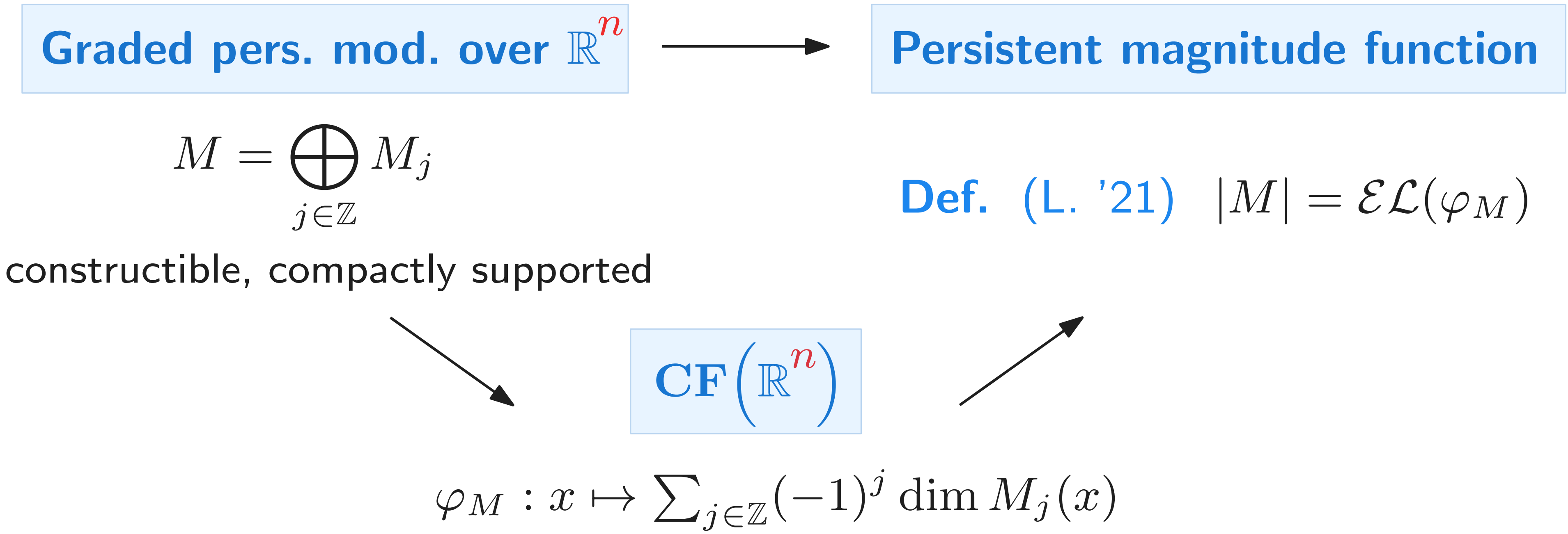
$$= \mathcal{E}\mathcal{L}(\varphi_M)(t)$$

$\mathbf{CF}(\mathbb{R})$

$$\varphi_M : t \mapsto \textstyle\sum_{j\in\mathbb{Z}} (-1)^j \dim M_j(t)$$

$$= \textstyle\sum_{j,k} (-1)^j \mathbf{1}_{[a_k^j, b_k^j)}$$

Multi-parameter Persistent magnitude



Graded pers. mod. over $\mathbb{R}^n$

$$M = \bigoplus_{j \in \mathbb{Z}} M_j$$

constructible, compactly supported

Persistent magnitude function

Def. (L. '21) $|M| = \mathcal{EL}(\varphi_M)$

$\mathbf{CF}\left(\mathbb{R}^n\right)$

$$\varphi_M : x \mapsto \sum_{j \in \mathbb{Z}} (-1)^j \dim M_j(x)$$

Multi-parameter Persistent magnitude

Graded pers. mod. over $\mathbb{R}^n$ → **Persistent magnitude function**

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Properties (L. '21)

- Compatibility with constructible operations (convolution, pushforward, ...)
- (Index theoretic formula) Let $f : X \to \mathbb{R}^n$ continuous subanalytic.

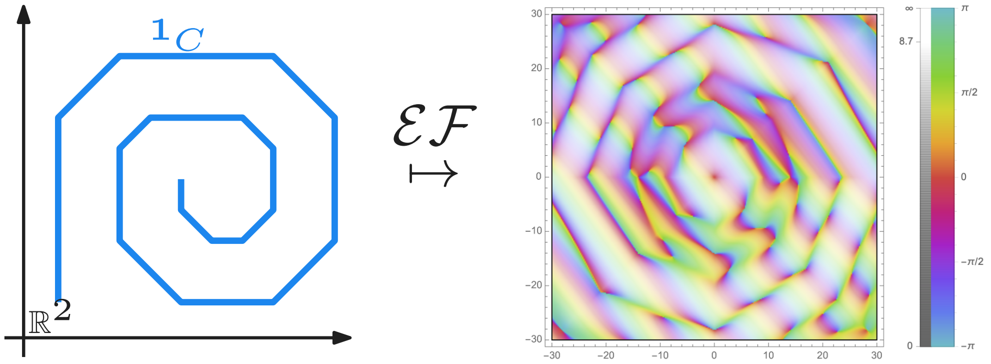
$$\forall \xi \in \mathbb{R}^n, \qquad |\mathrm{PH}(X, f)|(\xi) = \int_X e^{-\xi \circ f} \lfloor \mathrm{d}\chi \rfloor$$

Conclusion

more on **arXiv:2111.07829**

Hybrid transform

$$\mathrm{T}_\kappa[\varphi] : \begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{C} \\ \xi & \longmapsto & \displaystyle\int_{\mathbb{R}} \kappa(t)\xi_*\varphi(t)\,\mathrm{d}t \end{array}$$



To sum up : Integral transforms mixing $\int \cdot\,\mathrm{d}t$ and $\int \cdot\,\mathrm{d}\chi$.

Main results

- Output function is **regular** on PL-functions
- **Compatible** with constructible operations
- **Generalize** known invariants (e.g. **persistent magnitude**)
- **Left-inversion theorem** for Euler-Fourier
- Efficiently computable

  ↪ soon in **C++** and **Python** (joint with Oudot & Passe)

Conclusion more on **arXiv:2111.07829**

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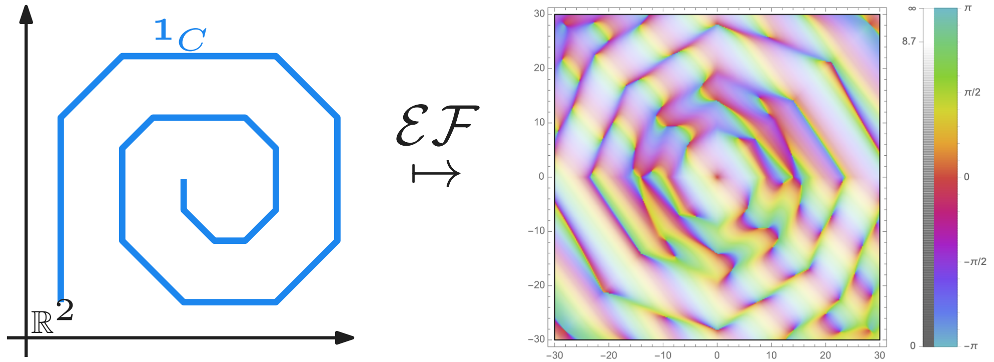

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Future work

- Stability with respect to $\varphi \in \mathrm{CF}(\mathbb{R}^n)$ (with F. Petit)
- Interpretability as invariant of persistent modules (with O. Hacquard)

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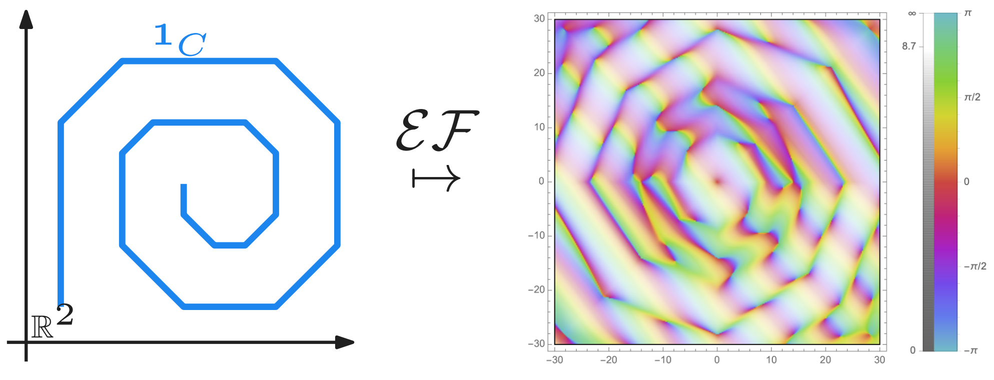

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Thank you !

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