# Analyse topologique de données

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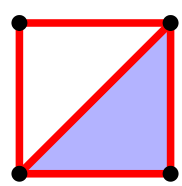
**Exercise 1** (Characterization of the convex hull)**.** Let $x_0, \dots, x_k \in \mathbb{R}^n$. Prove that:

$$\mathrm{Conv}(x_0, \dots, x_k) = \left\{ \sum_{i=0}^{k} \lambda_i x_i : \lambda_i \geq 0 \text{ and } \sum_{i=0}^{k} \lambda_i = 1 \right\}.$$

**Exercise 2** (Some combinatorics)**.** How many faces does a $k$-dimensional simplex have?

**Exercise 3** (Simplicial complexes)**.** Which of the following collections of simplices are simplicial complexes? $\mathcal{K}_1 = \{0\}$, $\mathcal{K}_2 = \{1\}$, $\mathcal{K}_3 = \{0, 1\}$, $\mathcal{K}_4 = \{0, 1, [0, 1]\}$, $\mathcal{K}_5 = \{0, [0, 1]\}$, $\mathcal{K}_6 = \{1, [0, 1]\}$, $\mathcal{K}_7$ given by the collection of vertices and edges of a triangle, $\mathcal{K}_8$ given by the collection of faces of a tetrahedra except its top dimensional face, $\mathcal{K}_9$ defined by the following figure:



where the white triangle is empty and the blue one if filled.

**Exercise 4** (Subcomplexes)**.** Let $\mathcal{K}$ and $\mathcal{K}'$ be two simplicial complexes. Prove the following assertions when they are true, or find a counterexample if not.

1. If $\mathcal{K}' \subset \mathcal{K}$, then $\mathcal{K} \setminus \mathcal{K}'$ is a subcomplex of $\mathcal{K}$.
2. The set $\mathcal{K} \cap K'$ is a simplicial complex.
3. The set $\mathcal{K} \cup \mathcal{K}'$ is a simplicial complex.

**Exercise 5** (PL-morphisms)**.** Show that the composition of two PL-morphisms is a PL-morphism.

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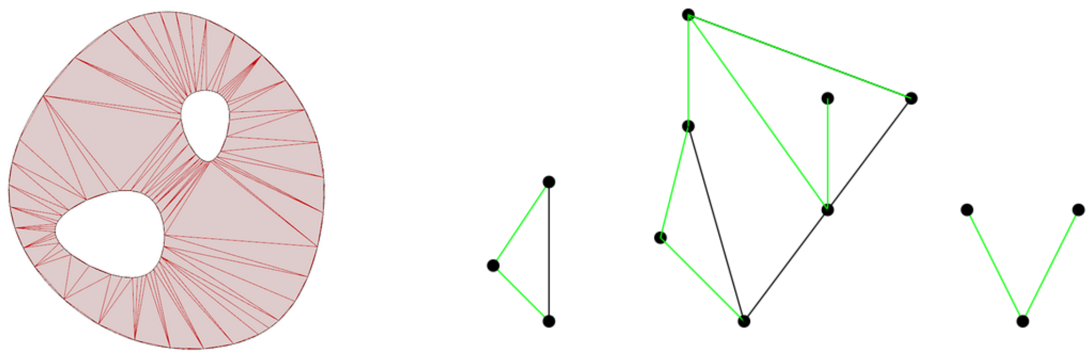
**Exercise 6** (Homology computations)**.** Let $\Bbbk$ be a field. Compute the homology with coefficients in $\Bbbk$ of the following simplicial complexes (see Exercise 3): $\mathcal{K}_1$, $\mathcal{K}_3$, $\mathcal{K}_4$, $\mathcal{K}_7$, $\mathcal{K}_8$. Can you guess the homology of $K_9$, and of the simplicial complexes $\mathcal{K}_{10}$ and $\mathcal{K}_{11}$ drawn in Figures 1a and 1b?



(a) $\mathcal{K}_{10}$

(b) $\mathcal{K}_{11}$

Figure 1: Two simplicial complexes

**Exercise 7** (Euler characteristic)**.** Let $\mathcal{K}$ be a simplicial complex. The *Euler characteristic* of $K$, denoted by $\chi(\mathcal{K})$, is the integer:

$$\chi(\mathcal{K}) = \sum_{\sigma \in \mathcal{K}} (-1)^{\dim(\sigma)}.$$

1. Show that if $\mathcal{K}$ and $\mathcal{K}'$ are two simplicial complexes such that $\mathcal{K} \cup \mathcal{K}'$ is a simplicial complex, then:
$$\chi(\mathcal{K} \cup \mathcal{K}') = \chi(\mathcal{K}) + \chi(\mathcal{K}') - \chi(\mathcal{K} \cap \mathcal{K}').$$
2. Prove the *Euler-Poincaré formula*:
$$\chi(\mathcal{K}) = \sum_{i \in \mathbb{Z}} (-1)^i \dim H_i(\mathcal{K}; \Bbbk).$$
3. Let $\mathcal{K}$ be a 1-dimensional simplicial complex. Use the Euler-Poincaré formula to compute the Betti numbers of $\mathcal{K}$ in terms of the number of vertices and edges of $\mathcal{K}$ and of path connected components of $|\mathcal{K}|$.

**Exercise 8** (Homology of compact surfaces[1])**.** We assume known the fact that the homology of a surface does not depend on a choice of triangulation. Use the triangulations of compact surfaces seen in the algebraic topology class to compute their homology.

[1]This exercise is harder.